Math 2102: Worksheet 9

1) Let $T : \mathbb{C}^6 \to \mathbb{C}^6$ be an operator with minimal polynomial $p_T(x) = (x-2)^2(x+1)^2$.

   (i) Determine all possible Jordan forms of T.

   (ii) Calculate the characteristic polynomial of each form in (i).

2) Consider the matrix:

$$A_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 1 & 0 \end{pmatrix}.$$

   (i) Calculate the Jordan canonical form of A_ϵ when $\epsilon \neq 0$;

   (ii) calculate the Jordan canonical form of A_ϵ when $\epsilon = 0$.

3) Let $v \in V$ and $w \in W$. Prove that $v \otimes w = 0$ if and only if $v = 0$ or $w = 0$.

4) Give an example of six distinct vectors $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}^3$ such that

$$v_1 \otimes w_1 + v_2 \otimes w_2 + v_3 \otimes w_3 = 0$$

   but none of $v_1 \otimes w_1$, $v_2 \otimes w_2$, or $v_3 \otimes w_3$ is a scalar multiple of another element in this list.

5) Let $\{v_1, \ldots, v_n\} \subset V$ be a list of linearly independent vectors. Consider $\{w_1, \ldots, w_n\} \subset W$ such that

$$v_1 \otimes w_1 + \cdots v_n \otimes w_n = 0.$$

   Prove that $w_1 = \cdots = w_n = 0$.

6) Assume that $\dim V > 1$ and $\dim W > 1$. Prove that $\{v \otimes w \mid v \in V,\ w \in W\} \neq V \otimes W$.

7) Let $\{v_1, \ldots, v_n\} \subset V$ and $\{w_1, \ldots, w_n\} \subset W$ be two sets of vectors such that

$$v_1 \otimes w_1 + \cdots v_n \otimes w_n = 0.$$

   Let $B : V \times W \to U$ be a bilinear functional into a vector space U. Prove that

$$B(v_1, w_1) + \cdots B(v_n, w_n) = 0.$$

8) Let $T : V_1 \to V_2$ and $S : W_1 \to W_2$ be two vector spaces.

   (i) Prove that there are linear maps $\mathrm{Id}_{V_1} \otimes S : V_1 \otimes W_1 \to V_1 \otimes W_2$, $T \otimes \mathrm{Id}_{W_1} : V_1 \otimes W_1 \to V_2 \otimes W_2$, and $T \otimes S : V_1 \otimes V_2 \to W_1 \otimes W_2$.

   (ii) Check that the following diagram commutes:

$$\begin{array}{ccc} V_1 \otimes V_2 & \xrightarrow{T \otimes \mathrm{Id}_{V_2}} & W_1 \otimes V_2 \\ {\scriptstyle \mathrm{Id}_{V_1} \otimes S}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_{W_1} \otimes S} \\ V_1 \otimes W_2 & \xrightarrow[T \otimes \mathrm{Id}_{W_2}]{} & W_1 \otimes W_2 \end{array}$$

   and that both of the composites equal $T \otimes S$.

(iii) Formulate and prove properties of $T \otimes S$ given properties of S and T. For instance, if S and T are injective what can you say about $S \otimes T$?

9) Let V and W be finite-dimensional vector spaces.

(i) Prove that the composition:

$$V^\vee \xrightarrow{c_V \otimes \mathrm{id}_{V^\vee}} V^\vee \otimes V \otimes V^\vee \xrightarrow{\mathrm{id}_{V^\vee} \otimes \mathrm{ev}'_V} V^\vee$$

is equal to the identity.

(ii) Let $T : V \to W$ and $T^\vee : W^\vee \to V^\vee$ be the map induced in the dual space. Check that the following diagram commutes:

$$\begin{array}{ccccccc} V^\vee & \xrightarrow{c_V \otimes \mathrm{id}_{V^\vee}} & V^\vee \otimes V \otimes V^\vee & \xrightarrow{T^\vee \otimes \mathrm{id}_{V \otimes V^\vee}} & V^\vee \otimes V \otimes V^\vee & \xrightarrow{\mathrm{id}_{V^\vee} \otimes \mathrm{ev}'_V} & V^\vee \\ \downarrow{\scriptstyle \mathrm{id}_{V^\vee}} & & & & & & \downarrow{\scriptstyle \mathrm{id}_{V^\vee}} \\ V^\vee & & & \xrightarrow[T^\vee]{} & & & V^\vee \end{array}$$

(iii) Consider the composite:

$$\tau(T) : \mathbb{F} \xrightarrow{c_V} V^\vee \otimes \xrightarrow{T^\vee \otimes \mathrm{id}_V} V^\vee \otimes V \xrightarrow{\mathrm{ev}_V} \mathbb{F}\ .$$

Prove that $\tau(T) = \mathrm{tr}(T)$.

10) Let V be a real inner product space and β a bilinear form on V. Prove that there exists an unique operator $T \in \mathcal{L}(V)$ such that

$$B(v, w) = \langle v, Tw \rangle \quad \text{for every } v, w \in V.$$

11) Find formulas for $\dim \mathcal{L}^2_{\mathrm{sym}}(V)$ and $\dim \mathcal{L}^2_{\mathrm{alt}}(V)$ in terms of $\dim V$.