Math 2102: Worksheet 8

1) Let $T \in \mathcal{L}(V)$ prove that

$$V = \operatorname{null} T \oplus \operatorname{range} T \Leftrightarrow \operatorname{null} T^2 = \operatorname{null} T.$$

2) Let $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ and $m \geq 1$ be an integer such that p_T is a multiple of $(z-\lambda)^m$. Prove that

$$\dim \operatorname{null}(T - \lambda \operatorname{Id}_V)^m \geq m.$$

3) (i) (This exercise does not use any of the new material we are learning, but it might be helpful for item (ii) below.) Let $T \in \mathcal{L}(V, W)$ where V is finite-dimensional, and $U \subseteq W$ be a subspace. Prove that $U' := \{v \in V \mid T(v) \in U\} \subseteq V$ is a subspace and that

$$\dim U' = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

(ii) Let $T \in \mathcal{L}(V)$ and $m \geq 1$. Prove that

$$\dim \operatorname{null} T^m \leq m \operatorname{null} T.$$

4) Assume that $T \in \mathcal{L}(V)$ is not nilpotent. Prove that $V = \operatorname{null} T^{\dim V - 1} \oplus \operatorname{range} T^{\dim V - 1}$.

5) Assume that $T \in \mathcal{L}(V)$ such that $\operatorname{null} T^{\dim V - 1} \neq \operatorname{null} T^{\dim V}$. Prove that T is nilpotent and that $\dim \operatorname{null} T^k = k$ for every $k \in \{0, 1, \ldots, \dim V\}$.

6) Let T be an operator on $\mathbb{F}^3$ whose matrix with respect to the standard basis is

$$\begin{pmatrix} -3 & 9 & 0 \\ -7 & 9 & 6 \\ 4 & 0 & 6 \end{pmatrix}.$$

Can you find a basis B_V of $\mathbb{F}^3$ such that $\mathcal{M}(T, B_V)$ is upper-triangular with only 0's on the diagonal?

7) Let $T \in \mathcal{L}(V)$ with eigenvalue λ and let d be the algebraic multiplicity of λ. Prove that

$$G(\lambda, T) = \operatorname{null}(T - \lambda)^d.$$

8) Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be given by $T(z_1, z_2, z_3, z_4) = (0, z_1, z_2, z_3)$. Find the minimal and characteristic polynomial of T.

9) Let $T : \mathbb{C}^6 \to \mathbb{C}^6$ be given by $T(z_1, z_2, z_3, z_4, z_5, z_6) = (0, z_1, z_2, 0, z_4, 0)$. Find the minimal and characteristic polynomial of T.

10) Assume that $\mathbb{F} = \mathbb{C}$. Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that the characteristic polynomial of P is $z^m(z-1)^n$, where $m = \dim \operatorname{null} P$ and $n = \dim \operatorname{range} P$.

11) Let $T \in \mathcal{L}(V)$ and λ be an eigenvalue of T. Explain why the following four numbers equal each other:

(a) the exponent of $(z - \lambda)$ in the minimal polynomial of T;

(b) the smallest positive number m such that $(T - \lambda)^m|_{G(\lambda, T)} = 0$;

(c) the smallest positive number m such that $\operatorname{null}(T-\lambda)^m = \operatorname{null}(T-\lambda)^{m+1}$;

(d) the smallest positive number m such that $\operatorname{range}(T-\lambda)^m = \operatorname{range}(T-\lambda)^{m+1}$.

12) Let $V = V_1 \oplus \cdots \oplus V_k$ and $T \in \mathcal{L}(V)$ such that each V_i is invariant under T. Prove that

$$c_T = \prod_{i=1}^{k} c_{T|_{V_i}},$$

i.e. the characteristic polynomial of T is the product of the characteristic polynomial of each of the restrictions $T|_{V_i} : V_i \to V_i$.

13) Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be given by $T(z, w) = (-w - z, 9w + 5z)$. Find a Jordan basis of $\mathbb{C}^2$ for T.

14) Find a basis of $\mathcal{P}_4(\mathbb{R})$ that is a Jordan basis for $D : \mathcal{P}_4(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ the differentiation operator, i.e. $D(p) := p'$.

15) Let $T \in \mathcal{L}(V)$ and consider $\{v_1, \ldots, v_n\}$ a basis of V that is a Jordan basis for T. Describe T^2 in this basis.

16) Prove that the trace is the only linear functional $\tau : \mathcal{L}(V) \to \mathbb{F}$ such that $\tau(ST) = \tau(TS)$ for all S and T and $\tau(\mathrm{Id}_V) = \dim V$.

17) Find $S, T \in \mathcal{L}(\mathcal{P}(\mathbb{F}))$ such that $ST - TS = \mathrm{Id}_{\mathcal{P}(\mathbb{F})}$.