Math 2102: Worksheet 8

1) Let $T \in \mathcal{L}(V)$ prove that

$$V = \operatorname{null} T \oplus \operatorname{range} T \Leftrightarrow \operatorname{null} T^2 = \operatorname{null} T.$$

Solution. *Assume that* $V = \operatorname{null} T \oplus \operatorname{range} T$. *Clearly,* $\operatorname{null} T \subseteq \operatorname{null} T^2$. *Let* $v \in \operatorname{null} T^2$, *then* $T(v) \in \operatorname{range} T \cap \operatorname{null} T$, *which implies that* $T(v) = 0$, *that is* $v \in \operatorname{null} T$.

Now assume that $\operatorname{null} T^2 = \operatorname{null} T$. *We first prove that* $\operatorname{range} T \cap \operatorname{null} T = \{0\}$. *Let* $v \in \operatorname{range} T \cap \operatorname{null} T$, *then* $v = T(u)$ *for some* u *and* $T(v) = T^2(u) = 0$. *Thus,* $u \in \operatorname{null} T^2 = \operatorname{null} T$, *which implies that* $v = 0$. *Now we prove that* $V = \operatorname{null} T \oplus \operatorname{range} T$. *We have:*

$$\operatorname{null} T \oplus \operatorname{range} T \subseteq V,$$

and the fundamental theorem of linear algebra gives that $\dim(\operatorname{null} T \oplus \operatorname{range} T) = \dim V$, *thus* $\operatorname{null} T \oplus \operatorname{range} T = V$.

2) Let $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ and $m \geq 1$ be an integer such that p_T is a multiple of $(z - \lambda)^m$. Prove that

$$\dim \operatorname{null}(T - \lambda \operatorname{Id}_V)^m \geq m.$$

Solution. *Let* $p_T(z) = (z - \lambda)^l q(z)$, *where* q *is a polynomial such that* $(z - \lambda)$ *is not a factor of* q. *Let* $G(\lambda, T) \subseteq V$ *be the generalized eigenspace of* λ *and consider* $U \subseteq V$ *some subspace such that* $G(\lambda, T) \oplus U = V$. *We claim that* $q(T|_U) = 0$. *Indeed, let* $u \in U$ *be non-zero then* $(T - \lambda)^l(u) \neq 0$, *since* $(T - \lambda)^l \circ q(T)(u) = 0$, *this implies that* $q(T)(u) = 0$.

We claim that $\operatorname{null}(T-\lambda)^{l-1} \neq \operatorname{null}(T-\lambda)^l$. *Indeed, if* $\operatorname{null}(T-\lambda)^l \subseteq \operatorname{null}(T-\lambda)^{l-1}$ *we notice that* $(T-\lambda)^{l-1}(v) = 0$ *for every* $v \in G(\lambda, T)$, *thus* $r(z) = (z-\lambda)^{l-1}q(z)$ *would be a minimal polynomial of* T, *which is a contradiction with* p_T *being the minimal polynomial. Then we have a chain of strict inclusions of subspaces*

$$\{0\} \subset \operatorname{null} T \subset \cdots \subset \operatorname{null} T^{l-1} \subset \operatorname{null} T^l,$$

which gives $\operatorname{null} T^k \geq k$ *for every* $k \in \{1, \ldots, l\}$. *In particular, we obtain the desired result.*

3) (i) (This exercise does not use any of the new material we are learning, but it might be helpful for item (ii) below.) Let $T \in \mathcal{L}(V, W)$ where V is finite-dimensional, and $U \subseteq W$ be a subspace. Prove that $U' := \{v \in V \mid T(v) \in U\} \subseteq V$ is a subspace and that

$$\dim U' = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

Solution. *Let* $u_1, u_2 \in U'$ *and* $a \in \mathbb{F}$ *we calculate*

$$T(au_1 + u_2) = aT(u_1) + T(u_2) \in U,$$

since $T(u_1), T(u_2) \in U$ *and* U *is a subspace.*

Notice that by definition $T|_{U'}$ *factors through* U, *i.e. we have a linear map* $S : U' \to U$ *such that*

$$\begin{array}{ccc} U' & \xrightarrow{\subset} & V \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ U & \xrightarrow[\subset]{} & V \end{array}$$

commutes. By applying the fundamental Theorem of Linear Algebra to S we obtain:

$$\dim U' = \dim \operatorname{null} S + \dim \operatorname{range} S.$$

We will be done if we prove that $\operatorname{range} S = \operatorname{range} T \cap U$ *and* $\operatorname{null} S = \operatorname{null} T$.
Let $x \in \operatorname{null} T$, since $T(x) = 0$, this gives that $x \in U'$, then $S(x) = T(x)$ which implies that $x \in \operatorname{null} S$. Now assume that $x \in \operatorname{null} S$, then $S(x) = T(x)$, which gives that $x \in \operatorname{null} T$.
Let $x \in \operatorname{range} S$, then there exists $y \in U'$ such that $T(y) = x$, thus $x \in U$ and $x \in \operatorname{range} T$. Let $x \in \operatorname{range} T \cap U$, then there exists $y \in V$ such that $T(y) = x$. Since $x \in U$, we ge that $y \in U'$, so $S(y) = T(y) = x$, i.e. $x \in \operatorname{range} S$.
This finishes the proof.

(ii) Let $T \in \mathcal{L}(V)$ and $m \geq 1$. Prove that

$$\dim \operatorname{null} T^m \leq m \dim \operatorname{null} T.$$

Solution. *We proceed by induction. The case $m = 1$ is clear.*
Assume the result holds for $m \geq 1$. Consider $U := \operatorname{null} T^m$ and $U' = \{v \in V \mid T(v) \in \operatorname{null} T^m\}$. Consider the diagram:

$$\begin{array}{ccc} U' & \xrightarrow{\subset} & V \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T^m} \\ U & \xrightarrow[\subset]{} & V \end{array},$$

where S is the restriction of T^m to U' which factors through U. We claim that $U' = \operatorname{null} T^{m+1}$. Indeed, let $x \in U'$, then $T(x) \in \operatorname{null} T^m$, so $T^{m+1} = 0$. Conversely, given $x \in \operatorname{null} T^{m+1}$, we notice that $T(x) \in \operatorname{null} T^m$, i.e. $x \in U'$. Thus, part (i) implies:

$$\dim \operatorname{null} T^{m+1} = \dim \operatorname{null} T + \dim(\operatorname{null} T^m \cap \operatorname{range} T) \leq \dim \operatorname{null} T + \dim \operatorname{null} T^m \leq (m+1) \dim \operatorname{null} T,$$

where in the last step we used the inductive hypothesis.

4) Assume that $T \in \mathcal{L}(V)$ is not nilpotent. Prove that $V = \operatorname{null} T^{\dim V - 1} \oplus \operatorname{range} T^{\dim V - 1}$.

Solution. *From Exercise 5) below we see that $\operatorname{null} T^{\dim V-1} = \operatorname{null} T^{\dim V}$, so $V = \operatorname{null} T^{\dim V-1} \oplus \operatorname{range} T^{\dim V}$. We claim that $\operatorname{null} T^{\dim V-1} \cap \operatorname{range} T^{\dim V-1} = \{0\}$. Indeed, let $u \in \operatorname{null} T^{\dim V-1} \cap \operatorname{range} T^{\dim V-1}$, then $u = T^{\dim V-1}(v)$ for some $v \in V$ and $T^{\dim V-1}(u) = 0$. Let $v = x + y$ where $x \in \operatorname{null} T^{\dim V-1}$ and $y \in \operatorname{range} T^{\dim V}$. Then $u = T^{\dim V-1+\dim V}(z)$ for some $z \in V$, which implies that $u \in \operatorname{range} T^{\dim V}$. Since $\operatorname{null} T^{\dim V-1} \cap \operatorname{range} T^{\dim V} = \{0\}$, we have that $u = 0$.*

From $\operatorname{null} T^{\dim V-1} \cap \operatorname{range} T^{\dim V-1} = \{0\}$ we conclude that $V = \operatorname{null} T^{\dim V-1} \oplus \operatorname{range} T^{\dim V-1}$ using the usual argument as in Exercise 1).

5) Assume that $T \in \mathcal{L}(V)$ such that $\operatorname{null} T^{\dim V-1} \neq \operatorname{null} T^{\dim V}$. Prove that T is nilpotent and that $\dim \operatorname{null} T^k = k$ for every $k \in \{0, 1, \dots, \dim V\}$.

Solution. *Notice that we have the chain of strict inclusions of subspaces:*

$$\{0\} \subset \operatorname{null} T \subset \cdots \operatorname{null} T^{\dim V-1} \subset \operatorname{null} T^{\dim V} \subseteq V.$$

This implies that $0 < \dim \operatorname{null} T < \cdots < \dim \operatorname{null} T^{\dim V-1} < \dim \operatorname{null} T^{\dim V}$. For each $k \geq 0$ there is a chain of k proper subspaces contained in $\operatorname{null} T^k$, this gives that $\dim \operatorname{null} T^k \geq k$ for every k. Notice that if $\dim T^k > k$ for some k, this implies that $\dim T^{\dim V} > \dim V$, which is a contradiction. Thus, we obtain that $\dim \operatorname{null} T^i = i$, which implies that $\operatorname{null} T^{\dim V} = V$ since it has the same dimension as V. In other words, we obtain that $T^{\dim V} = 0$, that is T is nilpotent.

6) Let T be an operator on $\mathbb{F}^3$ whose matrix with respect to the standard basis is

$$\begin{pmatrix} -3 & 9 & 0 \\ -7 & 9 & 6 \\ 4 & 0 & 6 \end{pmatrix}.$$

Can you find a basis B_V of $\mathbb{F}^3$ such that $\mathcal{M}(T, B_V)$ is upper-triangular with only 0's on the diagonal?

Solution. *No. We can simply compute the determinant of this matrix, which is* 432. *If we could find an upper-triangular form with only* 0*'s in the diagonal the determinant would be* 0. *Notice that we can use any matrix representation to compute the determinant, by Exercise 51 (ii) in the Lecture Notes.*

7) Let $T \in \mathcal{L}(V)$ with eigenvalue λ and let d be the algebraic multiplicity of λ. Prove that

$$G(\lambda, T) = \operatorname{null}(T - \lambda)^d.$$

Solution. *Recall that* $d := \dim G(\lambda, T)$. *By Theorem 7 (ii) in the Lecture Notes the restriction* $(T - \lambda)|_{G(\lambda,T)}$ *is nilpotent, so* $G(\lambda, T) = \operatorname{null}(T - \lambda)^m$ *for some* $m \geq 1$. *By Lemma 59 we know that* $m \leq \dim G(\lambda, T)$*; this proves the result.*

8) Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be given by $T(z_1, z_2, z_3, z_4) = (0, z_1, z_2, z_3)$. Find the minimal and characteristic polynomial of T.

Solution. *Notice that* $T^4 = 0$ *by* $T^3((1, 0, 0, 0)) \neq 0$*; so* $p_T(z) = z^4$. *Since* $c_T(z)$ *is a multiple of* $p_T(z)$ *and of dimension* 4, *we have that* $c_T(z) = z^4$ *as well.*

9) Let $T : \mathbb{C}^6 \to \mathbb{C}^6$ be given by $T(z_1, z_2, z_3, z_4, z_5, z_6) = (0, z_1, z_2, 0, z_4, 0)$. Find the minimal and characteristic polynomial of T.

Solution. *We notice that* $T^3 = 0$, *but* $T^2((1, 0, 0, 0, 0, 0)) \neq 0$*; so* T^3 *is the minimal polynomial. Notice that* 0 *is the only eigenvalue of* T. *By Corollary 26 (ii) we know that* $c_T(z) = (z - 0)^d$ *where* d *is the algebraic multiplicity of* 0. *Since* $G(0, T) = \mathbb{C}^6$, *we have* $c_T(z) = z^6$.

10) Assume that $\mathbb{F} = \mathbb{C}$. Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that the characteristic polynomial of P is $z^m(z - 1)^n$, where $m = \dim \operatorname{null} P$ and $n = \dim \operatorname{range} P$.

Solution. *First we check that the only eigenvalues of* P *are* 0 *and* 1. *Let* $\lambda \in \mathbb{F}$ *such that* $P(v) = \lambda v$ *for some non-zero* $v \in V$. *Then we have* $\lambda v = P(v) = P^2(v) = \lambda P(v) = \lambda^2 v$. *Since* v *is non-zero, we get* $\lambda = \lambda^2$. *This implies that* $\lambda = 0$ *or* $\lambda = 1$, *as claimed.*

By Corollary 26 (ii) we have that $c_T(z) = z^m(z - 1)^n$ *where* m *is the algebraic multiplicity of* 0 *and* n *the algebraic multiplicity of* 1. *Since we are over* $\mathbb{C}$ *we have*

$$V = G(0, P) \oplus G(1, P)$$

which gives $\dim V = \dim G(0, P) + \dim G(1, P)$. *Finally we claim that* $\operatorname{null} P = G(0, P)$ *and that* $\operatorname{range} P = G(1, P)$. *Indeed,* $G(0, P) = \operatorname{null} P^M$ *for* M *big enough, since* $P = P^2 = \cdots = P^M$ *we get* $G(0, P) = \operatorname{null} P$. *Similarly, notice that* $(P - \mathrm{Id}_V)^2 = P^2 - 2P + \mathrm{Id}_V = -(P - \mathrm{Id}_V)$, *which implies that* $\operatorname{null}(P - \mathrm{Id}_V)^N = \operatorname{null}(P - \mathrm{Id}_V)$ *for any* $N \geq 1$. *Thus,* $G(1, P) = \operatorname{null}(P - \mathrm{Id}_V)$. *Finally, notice that* $\operatorname{range} P = \operatorname{null}(P - \mathrm{Id}_V)$. *Indeed, let* $x \in \operatorname{range} P$, *then* $x = P(y)$ *for some* $y \in V$ *so* $P(x) - x = P^2(y) - P(y) = 0$. *If* $x \in \operatorname{null}(P - \mathrm{Id}_V)$ *then* $P(x) = x$, *so* $x \in \operatorname{range} P$.

This finishes the proof.

11) Let $T \in \mathcal{L}(V)$ and λ be an eigenvalue of T. Explain why the following four numbers equal each other:

(a) the exponent of $(z-\lambda)$ in the minimal polynomial of T;

(b) the smallest positive number m such that $(T-\lambda)^m|_{G(\lambda,T)} = 0$;

(c) the smallest positive number m such that $\operatorname{null}(T-\lambda)^m = \operatorname{null}(T-\lambda)^{m+1}$;

(d) the smallest positive number m such that $\operatorname{range}(T-\lambda)^m = \operatorname{range}(T-\lambda)^{m+1}$.

Solution. *The equivalence of (a) and (b) is in the solution of 2) above.*

The equivalence of (b) and (c) is proved similarly to the solution to Exercise 1. Noticing that $G(\lambda,T) = \operatorname{null}\,(T-\lambda)^m|_{G(\lambda,T)}$.

For (c) and (d) it is a similar argument to (b) and (c).

12) Let $V = V_1 \oplus \cdots \oplus V_k$ and $T \in \mathcal{L}(V)$ such that each V_i is invariant under T. Let $T_i : V_i \to V_i$ be the operator induced on V_i by $T|_{V_i}$ the restriction of T to V_i. Prove that

$$c_T = \prod_{i=1}^{k} c_{T_i},$$

i.e. the characteristic polynomial of T is the product of the characteristic polynomial of each of the operators $T_i : V_i \to V_i$.

Solution. *Let $G(\lambda,T)$ be a generalized eigenspace of V. We claim that*

$$G(\lambda,T) \cap V_i = G(\lambda,T_i).$$

Indeed, let $v \in G(\lambda,T_i)$, by definition we have that $v \in V_i$. Notice that $(T-\lambda\,\mathrm{Id}_V)^k v = (T_i - \lambda\,\mathrm{Id}_{V_i})^k(v)$ for any $k \geq 1$. So if $(T_i - \lambda\,\mathrm{Id}_{V_i})^k(v) = 0$ for some k then $(T-\lambda\,\mathrm{Id}_V)^k v = 0$ so $v \in G(\lambda,T)$. Conversely, if $v \in G(\lambda,T)\cap V_i$, notice that $(T-\lambda\,\mathrm{Id}_V)(v) = (T_i - \lambda\,\mathrm{Id}_{V_i})(v) \in V_i$. Thus, similarly if $(T-\lambda\,\mathrm{Id}_V)^k v = 0$ for some $k \geq 1$, then $(T_i-\lambda\,\mathrm{Id}_{V_i})^k(v) = 0$.

Moreover, we claim that

$$G(\lambda,T) = \oplus_i^k G(\lambda,T_i). \tag{1}$$

Indeed, it is clear that $G(\lambda,T_i)\cap G(\lambda,T_j) = \{0\}$. We claim that $\sum_{i=1}^k G(\lambda,T_i) = G(\lambda,T)$[1]. Let $v \in G(\lambda,T)$, then $v \in V_i$ for some $i \in \{1,\dots,k\}$, thus $T(v) = T_i(v) \in V_i$; so $v \in G(\lambda,T_i)$. This proves that $G(\lambda,T) \subseteq \sum_{i=1}^k G(\lambda,T_i)$. The other inclusion is clear from the definition.

Let $\{\lambda_1,\dots,\lambda_m\}$ be the list of all eigenvalues of T, notice that

$$c_{T_i}(z) = \prod_{j=1}^{m} (z-\lambda_j)^{\dim G(\lambda_j,T_i)},$$

where if $\dim G(\lambda_j,T_i) = 0$, then we only have the factor 1 in the expression above. Thus, we obtain:

$$\prod_{i=1}^{k} c_{T_i} = \prod_{i=1}^{k}\prod_{j=1}^{m}(z-\lambda_j)^{\dim G(\lambda_j,T_i)} = \prod_{j=1}^{m}(z-\lambda_j)^{\sum_{i=1}^k \dim(G(\lambda_j,T_i))} = \prod_{j=1}^{m}(z-\lambda_j)^{\dim G(\lambda,T)},$$

where in the last equation we used the equality $\dim G(\lambda,T) = \sum_{i=1}^k \dim(G(\lambda_j,T_i))$ obtained from the direct sum decomposition (1).

[1] Here we use the notation:

$$\sum_{i=1}^{k} G(\lambda,T_i) = G(\lambda,T_1) + \cdots + G(\lambda,T_k).$$

13) Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be given by $T(z, w) = (-w - z, 9w + 5z)$. Find a Jordan basis of $\mathbb{C}^2$ for T.

Solution. ***The following is what ChatGPT gave me as the answer. It is not complete– can you check what is correct and finish it?***

To find a Jordan basis for the transformation T, we first need to find the eigenvalues and corresponding eigenvectors. Let's start by finding the eigenvalues.

Let $v = (z, w)$ be an eigenvector of T with eigenvalue λ. Then, we have:

$T(v) = \lambda v$

$T(z, w) = (-w - z, 9w + 5z) = (\lambda z, \lambda w)$

Now we have a system of equations:

$-w - z = \lambda z \; 9w + 5z = \lambda w$

Let's solve this system of equations to find the eigenvalues λ. We can rewrite the first equation as:

$z = -w - \lambda z$

Now substitute this expression for z into the second equation:

$9w + 5(-w - \lambda z) = \lambda w$

$9w - 5w - 5\lambda z = \lambda w$

$4w = (5\lambda - 1)z$

Now we consider two cases:

1) If $z \neq 0$, we can write:

$w = \frac{5\lambda-1}{4} z$

Plugging this back into the equation for z:

$z = -\frac{5\lambda-1}{4} z - \lambda z$

$1 + \frac{5\lambda-1}{4} = \lambda$

Solving for λ, we get:

$\lambda = 1$

2) If $z = 0$, then from the equation for w:

$4w = (5\lambda - 1)0$

$4w = 0$

$w = 0$

In this case, both z and w are zero, which means the eigenvector is the zero vector, which is not a valid eigenvector.

So, we have one eigenvalue $\lambda = 1$. Now let's find the corresponding eigenvector:

$-w - z = z$

$w = -2z$

So the eigenvector is $v = (z, -2z)$. We can take a basis vector, for example, $v_1 = (1, -2)$.

Since we have only one eigenvector and it corresponds to the eigenvalue $\lambda = 1$, the Jordan form of T is a single Jordan block with eigenvalue 1. The Jordan basis is given by the eigenvector $v_1 = (1, -2)$.

14) Find a basis of $\mathcal{P}_4(\mathbb{R})$ that is a Jordan basis for $D : \mathcal{P}_4(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ the differentiation operator, i.e. $D(p) := p'$.

Solution. ***Also courtesy of ChatGPT.*** *To find a Jordan basis for the differentiation operator D acting on the vector space $\mathcal{P}_4(\mathbb{R})$, we need to find the eigenvalues and corresponding eigenvectors/generalized eigenvectors of D.*

Recall that an eigenvector of D is a polynomial $p(x)$ such that:

$D(p(x)) = \lambda p(x)$

where λ is the eigenvalue. Since $D(p(x))$ is the derivative of $p(x)$, the equation becomes:

$p'(x) = \lambda p(x)$

Now we consider different cases for the degree of $p(x)$:

1) If the degree of $p(x)$ is 0, i.e., $p(x) = c$ for some constant c, then $p'(x) = 0$. In this case, the eigenvalue is $\lambda = 0$ and the corresponding eigenvector is the constant polynomial $p_0(x) = 1$.

2) If the degree of $p(x)$ is greater than 0, then the equation $p'(x) = \lambda p(x)$ has no nontrivial solutions for $\lambda \neq 0$. This is because the degree of $p'(x)$ is strictly less than the degree of $p(x)$, so they cannot be proportional unless $p'(x) = 0$. However, the only polynomials with zero derivatives are constant polynomials, which we have already considered in case (1).

Thus, the only eigenvalue of D is $\lambda = 0$. Now we need to find the generalized eigenvectors for this eigenvalue. For this, we need to find the polynomials $p(x)$ that satisfy the equation $(D-\lambda I)^k(p(x)) = 0$ for some positive integer k. Since $\lambda = 0$, we have:

$(D - 0I)^k(p(x)) = D^k(p(x)) = 0$

Let's find the smallest k such that $D^k(p(x)) = 0$ for each possible degree of $p(x)$:

- For degree 0: $D^1(p(x)) = 0$. We already found the eigenvector $p_0(x) = 1$. - For degree 1: $D^2(p(x)) = 0$. A basis vector is $p_1(x) = x$. - For degree 2: $D^3(p(x)) = 0$. A basis vector is $p_2(x) = \frac{1}{2}x^2$. - For degree 3: $D^4(p(x)) = 0$. A basis vector is $p_3(x) = \frac{1}{6}x^3$. - For degree 4: $D^5(p(x)) = 0$. A basis vector is $p_4(x) = \frac{1}{24}x^4$.

Thus, a Jordan basis for D acting on $\mathcal{P}_4(\mathbb{R})$ is given by the set of polynomials:

$\{p_0(x) = 1, p_1(x) = x, p_2(x) = \frac{1}{2}x^2, p_3(x) = \frac{1}{6}x^3, p_4(x) = \frac{1}{24}x^4\}$.

15) Let $T \in \mathcal{L}(V)$ and consider $\{v_1, \dots, v_n\}$ a basis of V that is a Jordan basis for T. Describe T^2 in this basis.

Solution. ***This is also from ChatGPT.*** *Since $\{v_1, \dots, v_n\}$ is a Jordan basis for T, we know that the matrix representation of T with respect to this basis is in Jordan normal form. Let J_T be the matrix representation of T in this basis, so that J_T is a block diagonal matrix where each block is a Jordan block.*

Now, we want to find the matrix representation of T^2 with respect to the same basis. To do this, we simply compute the square of the matrix J_T. Since the Jordan blocks are independent and diagonal, we can compute the square of each block separately and then combine them into the matrix representation of T^2.

Let $J_{\lambda,k}$ be a Jordan block corresponding to the eigenvalue λ and of size $k \times k$. The matrix form of this block is:

$$J_{\lambda,k} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

To find $(J_{\lambda,k})^2$, we can compute the matrix product:

$(J_{\lambda,k})^2 = J_{\lambda,k} \cdot J_{\lambda,k}$

Notice that when multiplying the matrices, the only non-zero entries will be in the diagonal and the superdiagonal. The diagonal entries are the eigenvalue squared, and the superdiagonal entries are the product of the eigenvalue and the superdiagonal entry in the original matrix. Thus, we have: ***Do you agree with this? I tried the case k=2 and was almost convinced...***

$$(J_{\lambda,k})^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 0 & \cdots & 0 \\ 0 & \lambda^2 & 2\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^2 \end{pmatrix}$$

Now, since J_T is a block diagonal matrix with Jordan blocks, its square J_{T^2} will also be a block diagonal matrix with the squares of the corresponding Jordan blocks. Therefore, the matrix representation of T^2 with respect to the Jordan basis is given by J_{T^2}, which is a block diagonal matrix with blocks $(J_{\lambda,k})^2$ as described above.

16) Prove that the trace is the only linear functional $\tau : \mathcal{L}(V) \to \mathbb{F}$ such that $\tau(ST) = \tau(TS)$ for all S and T and $\tau(\mathrm{Id}_V) = \dim V$.

Solution. *A more general result is proved in Lemma 70 in the Lecture Notes. Notice that you can probably find a simpler proof here since we are only considering S and T operators on the same vector space.*

17) Find $S, T \in \mathcal{L}(\mathcal{P}(\mathbb{F}))$ such that $ST - TS = \mathrm{Id}_{\mathcal{P}(\mathbb{F})}$.

Solution. *Let $S : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be $S(p(x)) = p'(x)$ and $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be $T(p(x)) = xp(x)$. Now we compute:*

$$ST(p(x)) = (xp(x))' = p(x) + xp'(x) \quad \textit{and} \quad TS(p(x)) = x(p(x))' = xp'(x).$$

Thus, $(ST - TS)(p(x)) = p(x)$ for every $p(x) \in \mathcal{P}(\mathbb{F})$, thus $ST - TS = \mathrm{Id}_{\mathcal{P}(\mathbb{F})}$.