Math 2102: Worksheet 7

1) Let $\{e_1, \ldots, e_n\}$ be a set of vectors in V such that $\|v_i\| = 1$ for every $1 \leq i \leq n$. Suppose that

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle| \qquad \text{for every } v \in V.$$

Prove that $\langle v_i, v_j \rangle = 0$ for $i \neq j$, i.e. $\{e_1, \ldots, e_n\}$ is an orthonormal set.

2) Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V. Consider $\{f_1, \ldots, f_n\}$ a dual basis of $V^\vee := \mathcal{L}(V, \mathbb{F})$ the dual space of V. For each $f_i \in V^\vee$, the Riesz representation theorem (Proposition 7), shows that there exist unique $v_i \in V$ such that $f_i(u) = \langle u, v_i \rangle$. Prove that $v_i = e_i$.

3) Let $T : V \to W$ be a linear map between inner product spaces.

   (i) Given orthonormal bases $\{e_1, \ldots, e_n\}$ of V and $\{f_1, \ldots, f_m\}$ of W prove that

   $$\sum_{i=1}^{n} \|Te_i\|^2 = \sum_{j=1}^{m} \|T^* f_j\|\,.$$

   (ii) Prove that T is injective if and only if T^* is surjective.

   (iii) Prove that T is surjective if and only if T^* is injective.

   (iv) $\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$.

   (v) $\dim \operatorname{range} T = \dim \operatorname{range} T$.

4) Define an inner product on $\mathcal{P}_2(\mathbb{R})$ by $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$. Let the operator $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ be defined as:

$$T(ax^2 + bx + c) = bx.$$

   (i) Show that T is not self-adjoint.

   (ii) Calculate the matrix representing T on the basis $1, x, x^2$. Notice this matrix is equal to its conjugate transpose. Why is this not a contradiction with (i)?

5) Let $T \in \mathcal{L}(V)$ be a normal operator.

   (i) Prove that $\operatorname{range} T^k = \operatorname{range} T$ for every $k \geq 1$.

   (ii) Prove that $\operatorname{null} T^k = \operatorname{null} T$ for every $k \geq 1$.

   (iii) Let $\lambda \in \mathbb{F}$, prove p_T, the minimal polynomial of T, is not a multiple of $(x - \lambda)^2$.

6) Let $T : V \to V$ be a normal operator on a complex vector space.

   (i) Prove that T is self-adjoint if and only if all of the eigenvalues of T are real.

   (ii) Prove that $T = -T^*$ if and only if all of the eigenvalues of T are purely imaginary, i.e. complex numbers with 0 real part.

7) Prove or give a counter-example. Every diagonalizable operator $T \in \mathcal{L}(\mathbb{C}^3)$ is normal.

8) Let $T \in \mathcal{L}(V)$ be an operator on a finish-dimensional inner product space.

(i) Assume that $\mathbb{F} = \mathbb{R}$. Prove that T is self-adjoint if and only if (a) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$ for distinct eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ and (b) $\langle v_i, v_j \rangle = 0$ for $v_i \in E(\lambda_i, T)$ and $E(\lambda_j, T)$ for $i \neq j$.

(ii) Assume that $\mathbb{F} = \mathbb{C}$. Prove that T is normal if and only if (a) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$ for distinct eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ and (b) $\langle v_i, v_j \rangle = 0$ for $v_i \in E(\lambda_i, T)$ and $E(\lambda_j, T)$ for $i \neq j$.

9) Give an example of $T : V \to V$ on a real inner product space such that there are real numbers $b, c \in \mathbb{R}$ such that

$$b^2 < 4c \quad \text{and} \quad T \text{ is not invertible.}$$

In particular, this shows that we can not relax the assumption that T is self-adjoit in the real spectral theorem.

10) Let $T : V \to V$ be a self-adjoint operator and $U \subseteq V$ a subspace invariant under T.

(i) Prove that $U^\perp$ is invariant under T.

(ii) Prove that $T|_U \in \mathcal{L}(U)$ is self-adjoint.

(iii) Prove that $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

11) Let $T : V \to V$ be a normal operator and $U \subseteq V$ a subspace invariant under T.

(i) Prove that $U^\perp$ is invariant under T.

(ii) Prove that U is invariant under T^*.

(iii) Prove that $(T|_U)^* = (T^*)|_U$.

(iv) Prove that $T|_U \in \mathcal{L}(U)$ and $T|_{U^\perp}$ are normal operators.

(v) (Extra) Use the items above to give, yet another, proof of the complex spectral theorem.

12) Let $T \in \mathcal{L}(V)$ be self-adjoint, $\lambda \in \mathbb{F}$ and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \epsilon$. Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.