Math 2102: Worksheet 7
Solutions

1) Let $\{e_1, \ldots, e_n\}$ be a set of vectors in V such that $\|e_i\| = 1$ for every $1 \leq i \leq n$. Suppose that

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2 \qquad \text{for every } v \in V.$$

Prove that $\langle e_i, e_j \rangle = 0$ for $i \neq j$, i.e. $\{e_1, \ldots, e_n\}$ is an orthonormal set.

Solution. *By considering $v = e_j$ we have:*

$$\|e_j\|^2 = \|e_j\|^4 + \sum_{i \neq j} |\langle e_j, e_i \rangle|^2.$$

Thus, $\sum_{i \neq j} |\langle e_j, e_i \rangle|^2 = 0$. Since each term $|\langle e_j, e_i \rangle| \geq 0$, we obtain $\langle e_j, e_i \rangle = 0$ for $i \neq j$.

2) Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V. Consider $\{f_1, \ldots, f_n\}$ a dual basis of $V^\vee := \mathcal{L}(V, \mathbb{F})$ the dual space of V. For each $f_i \in V^\vee$, the Riesz representation theorem (Proposition 7), shows that there exist unique $v_i \in V$ such that $f_i(u) = \langle u, v_i \rangle$. Prove that $v_i = e_i$.

Solution. *Notice that by definition we have $f_i(e_j) = \delta_{i,j}$, thus we obtain:*

$$\langle e_j, v_i \rangle = \delta_{j,i}.$$

Since any $v_i = \sum_{j=1}^{n} a_{i,j} e_j$ we see that $\delta_{k,i} = \langle e_k, v_i \rangle = a_{k,i}$, which finishes the proof.

3) Let $T : V \to W$ be a linear map between inner product spaces.

(i) Given orthonormal bases $\{e_1, \ldots, e_n\}$ of V and $\{f_1, \ldots, f_m\}$ of W prove that

$$\sum_{i=1}^{n} \|Te_i\|^2 = \sum_{j=1}^{m} \|T^* f_j\|^2.$$

Solution. *We calculate*

$$\begin{aligned}
\sum_{i=1}^{n} \|Te_i\|^2 &= \sum_{i=1}^{n} \sum_{j=1}^{m} |\langle Te_i, f_j \rangle|^2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle Te_i, f_j \rangle \langle f_j, Te_i \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle e_i, T^* f_j \rangle \langle T^* f_j, e_i \rangle \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} |\langle T^* f_j, e_i \rangle|^2 \\
&= \sum_{j=1}^{m} \|T^* f_j\|^2,
\end{aligned}$$

where in the first line and between the last two lines we used Lemma 44 (ii). The other equalities follow from the definition of T^* and inner product.

(ii) Prove that T is injective if and only if T^* is surjective.

(iii) Prove that T is surjective if and only if T^* is injective.

(iv) $\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$.

(v) $\dim \operatorname{range} T = \dim \operatorname{range} T^*$.

4) Define an inner product on $\mathcal{P}_2(\mathbb{R})$ by $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$. Let the operator $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ be defined as:

$$T(ax^2 + bx + c) = bx.$$

(i) Show that T is not self-adjoint.

(ii) Calculate the matrix representing T on the basis $1, x, x^2$. Notice this matrix is equal to its conjugate transpose. Why is this not a contradiction with (i)?

5) Let $T \in \mathcal{L}(V)$ be a normal operator.

(i) Prove that $\operatorname{range} T^k = \operatorname{range} T$ for every $k \geq 1$.

Solution. *By Lemma 50 in the Lecture notes we have:*

$$\operatorname{range} T^k = (\operatorname{null}(T^*)^k)^\perp.$$

Notice that if T is normal, then T^* is also normal; since $(T^*)^* = T$. Thus, part (ii) below implies that $\operatorname{null}(T^*)^k = \operatorname{null} T^*$, which gives that

$$\operatorname{range} T^k = (\operatorname{null}(T^*)^k)^\perp = (\operatorname{null} T^*)^\perp = \operatorname{range} T.$$

(ii) Prove that $\operatorname{null} T^k = \operatorname{null} T$ for every $k \geq 1$.

Solution. *Notice that if S is a self-adjoint operator, then we have $S^k v = 0$ for any $k \geq 1$, implies $Sv = 0$. Indeed, we have $0 = \langle S^k v, S^{k-2} v \rangle = \langle S^{k-1} v, S^{k-1} v \rangle = \left\| S^{k-1} v \right\|^2 = 0$, which implies $S^{k-1} v = 0$.*

It is clear that $\operatorname{null} T \subseteq \operatorname{null} T^k$. Assume that $v \in \operatorname{null} T^k$, then $(T^*T)^k v = (T^*)^k T^k v = 0$. Since T^*T is self-adjoint, the previous paragraph implies that $T^*Tv = 0$. Now we have: $\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = \|Tv\|^2 = 0$, thus $Tv = 0$.

(iii) Let $\lambda \in \mathbb{F}$, prove that p_T, the minimal polynomial of T, is not a multiple of $(x - \lambda)^2$.

Solution. *The same argument above proves that $\operatorname{null}(T - \lambda)^k = \operatorname{null}(T - \lambda)$. Thus, if $(x - \lambda)^2$ is a factor of the minimal polynomial, we obtain a contradiction with p_T being of minimal degree.*

6) Let $T : V \to V$ be a normal operator on a complex vector space.

(i) Prove that T is self-adjoint if and only if all of the eigenvalues of T are real.

Solution. *By the spectral Theorem we have that T is diagonalizable in some orthonormal basis with the entries $\{\lambda_1, \ldots, \lambda_n\}$ on the diagonal. Thus, T^* has*

$$\{\overline{\lambda_1}, \ldots, \overline{\lambda_n}\}$$

on the diagonal. If $T = T^*$ we obtain that $\lambda_i = \overline{\lambda_i}$ for each $i \in \{1, \ldots, n\}$. Since the eigenvalues of T are a subset of $\{\lambda_1, \ldots, \lambda_n\}$ we obtain the claim.

(ii) Prove that $T = -T^*$ if and only if all of the eigenvalues of T are purely imaginary, i.e. complex numbers with 0 real part.

Solution. *A similar argument as in (i) gives that* $\lambda_i = -\lambda_i$ *for every* $i \in \{1, \ldots, n\}$*. So the claim follows.*

7) Prove or give a counter-example. Every diagonalizable operator $T \in \mathcal{L}(\mathbb{C}^3)$ is normal.

Solution. *Let* $T(x, y, z) = (-x + 2y, y, z)$ *then one can check that*

$$\mathcal{M}(T, B_V) = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for B_V *the standard basis. However, we have:*

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $T^*T \neq TT^*$*. But on the basis* $B_W = \{e_1 + e_2, e_1, e_3\}$ *we have that*

$$\mathcal{M}(T, B_W) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so T *is diagonalizable.*

8) Let $T \in \mathcal{L}(V)$ be an operator on a finish-dimensional inner product space.

   (i) Assume that $\mathbb{F} = \mathbb{R}$. Prove that T is self-adjoint if and only if (a) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$ for distinct eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ and (b) $\langle v_i, v_j \rangle = 0$ for $v_i \in E(\lambda_i, T)$ and $E(\lambda_j, T)$ for $i \neq j$.

   (ii) Assume that $\mathbb{F} = \mathbb{C}$. Prove that T is normal if and only if (a) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$ for distinct eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ and (b) $\langle v_i, v_j \rangle = 0$ for $v_i \in E(\lambda_i, T)$ and $E(\lambda_j, T)$ for $i \neq j$.

9) Give an example of $T : V \to V$ on a real inner product space such that there are real numbers $b, c \in \mathbb{R}$ such that

$$b^2 < 4c \quad \text{and} \quad T^2 + bT + c\,\mathrm{Id}_V \text{ is not invertible.}$$

In particular, this shows that we can not relax the assumption that T is self-adjoit in the real spectral theorem.

Solution. *Consider* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $T(x, y) = (-y, x)$*. Notice that*

$$T^2 + \mathrm{Id}_V = 0.$$

10) Let $T : V \to V$ be a self-adjoint operator and $U \subseteq V$ a subspace invariant under T.

   (i) Prove that $U^\perp$ is invariant under T.

   (ii) Prove that $T|_U \in \mathcal{L}(U)$ is self-adjoint.

   (iii) Prove that $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

11) Let $T : V \to V$ be a normal operator and $U \subseteq V$ a subspace invariant under T.

(i) Prove that $U^\perp$ is invariant under T.

Solution. *Let $B_U = \{u_1, \ldots, u_k\}$ be an orthonormal basis of U, which can be extended to $\{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ an orthonormal basis of V. Let*

$$\mathcal{M}(T, B_V) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

be the block form of the matrix representing T. Notice that the lower left block vanishes since U is invariant under T. We need to prove that B also vanishes.
Since T is normal we have $\|Tv\| = \|T^*v\|$ for every $v \in V$, in particular, we have $\|Tu_i\| = \|T^*u_i\|$ for every $i \in \{1, \ldots, k\}$, which implies that

$$\sum_{i=1}^{k} \|Tu_i\|^2 = \sum_{i=1}^{k} \|T^*u_i\|^2 .$$

Notice that $\|Tu_i\|^2 = \sum_{j=1}^{k} |A_{ji}|^2$, whereas $\|Tu_i\|^2 = \sum_{j=1}^{k} |A^\dagger_{ji}|^2 + \sum_{j=1}^{n-k} |B^\dagger_{ji}|^2$. Since $|A^\dagger_{ji}| = |A_{ij}|$, we obtain:

$$\begin{aligned} \sum_{i=1}^{k}\sum_{j=1}^{k} |A_{ji}|^2 &= \sum_{i=1}^{k} \|Tu_i\|^2 \\ &= \sum_{i=1}^{k} \|T^*u_i\|^2 \\ &= \sum_{i=1}^{k}(\sum_{j=1}^{k} |A^\dagger_{ji}|^2 + \sum_{j=1}^{n-k} |B^\dagger_{ji}|^2) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} |A_{ij}|^2 + \sum_{i=1}^{k}\sum_{j=1}^{n-k} |B^\dagger_{ji}|^2). \end{aligned}$$

This implies that $B_{ij} = B^\dagger_{ji} = 0$ for every $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n-k\}$. Thus, B vanishes.

(ii) Prove that U is invariant under T^*.

Solution. *Let $u \in U$ and $v \in U^\perp$, then we have:*

$$\langle T^*u, v\rangle = \langle u, Tv\rangle = 0,$$

since $Tv \in U^\perp$, this implies that $T^*(U) \subseteq (U^\perp)^\perp$. However, since V is finite-dimensional, Lemma 47 from the Lecture Notes gives that $(U^\perp)^\perp = U$.

(iii) Prove that $(T|_U)^* = (T^*)|_U$.

Solution. *Notice that $T^*|_U : U \to V$ is always well-defined, and in the basis taken in (i) it is represented as:*

$$\mathcal{M}(T^*|_U : U \to V, B_U, B_V) = \begin{pmatrix} A^\dagger \\ B^\dagger \end{pmatrix},$$

where $B = 0$, implies $B^\dagger = 0$, which implies that $T^*|_U$ factors as $\overline{T^*|_U} : U \to U$. The claim that $\overline{T^*|_U} = (T|_U)^*$ now is clear from the matrix representation.

(iv) Prove that $T|_U \in \mathcal{L}(U)$ and $T|_{U^\perp}$ are normal operators.

Solution. *This amounts to noticing that*

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} A^\dagger & 0 \\ 0 & C^\dagger \end{pmatrix} = \begin{pmatrix} A^\dagger & 0 \\ 0 & C^\dagger \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$

implies $AA^\dagger = A^\dagger A$, *i.e.* $T|_U \, (T|_U)^* = (T|_U)^* \, T|_U$ *and* $BB^\dagger = B^\dagger B$, *i.e.* $T|_{U^\perp} \, (T|_{U^\perp})^* = (T|_{U^\perp})^* \, T|_{U^\perp}$.

(v) (Extra) Use the items above to give, yet another, proof of the complex spectral theorem.

12) Let $T \in \mathcal{L}(V)$ be self-adjoint, $\lambda \in \mathbb{F}$ and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \epsilon$. Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.