Math 2102: Worksheet 6

1) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be given by $T(z_1, z_2, z_3) = (2z_1 + z_2 + 3z_3, 2z_2 + 2z_3, 3z_3)$. Determine if T is diagonalizable or not.

2) Suppose that V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ denote the nonzero eigenvalues of T. Prove that
$$\sum_{i=1}^{n} \dim E(\lambda_i, T) \leq \dim \operatorname{range} T.$$

3) Consider the inner product on $\mathcal{P}_2(\mathbb{R})$ given by $\langle p, q \rangle := \int_0^1 pq$.
   (i) Apply the Gram–Schmidt procedure to $\{1, x, x^2\}$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.
   (ii) Find the matrix representing differentiation on the basis $B = \{1, x, x^2\}$ and then on the basis obtained in (i). Check that both of these are upper-triangular. This is an example of Lemma 45 in the Lecture Notes.
   (iii) Consider the linear functional:
$$\begin{aligned} \lambda : \mathcal{P}_2(\mathbb{R}) &\to \mathbb{R} \\ p &\mapsto \int_0^1 \cos(\pi x) p(x) dx. \end{aligned}$$
   Determine $q \in \mathcal{P}_2(\mathbb{R})$ such that $\lambda(-) = \langle -, q \rangle$, which exists by Riesz representation theorem.

4) (Gershgorin Disk Theorem) Let $T : V \to V$ be a linear operator and $B_V = \{v_1, \ldots, v_n\}$ be a basis of V. Let $(a_{i,j})_{1 \leq i,j \leq n} = \mathcal{M}(T, B_V)$ denote the matrix representing T in the basis B_V. For each $i \in \{1, \ldots, n\}$ we define the *ith Gershgorin disk* to be:
$$D_i := \{z \in \mathbb{F} \mid |z - a_{i,i}| \leq \sum_{j \neq i} |a_{i,j}|\}.$$
   (i) Prove that each eigenvalue λ of T belongs to D_i for some $i \in \{1, \ldots, n\}$.
   (ii) Assume that $\sum_{j \neq i} |a_{i,j}| < |a_{i,i}|$ for every $i \in \{1, \ldots, n\}$. Prove that T is invertible. Can you give an example of a matrix such that this results allows you to deduce that it is invertible.
   (iii) Let $D_i^{\text{col.}} := \{z \in \mathbb{F} \mid |z - a_{i,i}| \leq \sum_{j \neq i} |a_{j,i}|\}$, i.e. one change the definition of disks to compare the value of the diagonal entries with the other values in its column. Check that the same statement as in (i) holds for $D_i^{\text{col.}}$.