Math 2102: Worksheet 6
Solutions

1) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be given by $T(z_1, z_2, z_3) = (2z_1 + z_2 + 3z_3, 2z_2 + 2z_3, 3z_3)$. Determine if T is diagonalizable or not.

   Solution. *Notice that the eigenvalues of T are 2 and 3, since it is in upper-triangular form and we can simply read them from the diagonal entries. Thus, the minimal polynomial of T is either $a(z) = (z-2)(z-3)$ or $b(z) = b(z-2)^2(z_3)$. However, we see that*

   $$(T-2\,\mathrm{Id}_V)(T-3\,\mathrm{Id}_V)((0,1,0)) = (T-2\,\mathrm{Id}_V)((2,-2,0)) = (4-2,-4,0)-2(2,-2,0) = (-2,0,0) \neq 0.$$

   So the minimal polynomial of T is $(z_2)^2(z-3)$ and by Lemma 38 we see that T is not diagonalizable.

2) Suppose that V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ denote the distinct nonzero eigenvalues of T. Prove that

   $$\sum_{i=1}^{n} \dim E(\lambda_i, T) \leq \dim \operatorname{range} T.$$

   Solution. *Let $S \subset \mathbb{F}$ be the set of all eigenvalues of T. By Lemma 37, we have that*

   $$\sum_{s\in S} E(\lambda_s, T) = \oplus_{s\in S} E(\lambda_s, T) \subseteq V.$$

   Thus, $\sum_{s\in S} \dim E(\lambda_s, T) = \dim(\sum_{s\in S} E(\lambda_s, T)) \leq \dim V$, where the first equality follows from Exercise 20 (i) in the Lecture Notes. Now, we notice that $E(0, T) = \operatorname{null} T$, thus we get:

   $$\sum_{i=1}^{n} \dim E(\lambda_i, T) + \dim \operatorname{null} T = \sum_{s\in S} \dim E(\lambda_s, T) \leq \dim V.$$

   So $\sum_{i=1}^{n} \dim E(\lambda_i, T) \leq \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$, by the Fundamental Theorem of Linear Algebra.

3) Consider the inner product on $\mathcal{P}_2(\mathbb{R})$ given by $\langle p, q\rangle := \int_0^1 pq$.

   (i) Apply the Gram–Schmidt procedure to $\{1, x, x^2\}$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

   Solution. *To apply the Gram-Schmidt procedure to the set $\{1, x, x^2\}$, we'll find the orthogonal projections and normalize them to obtain an orthonormal basis.*

   1. Start with the first vector $v_1 = 1$. Normalize it to get the first orthonormal basis vector:

   $$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\langle 1,1\rangle}} = \frac{1}{\sqrt{\int_0^1 1\cdot 1dx}} = \frac{1}{\sqrt{1}} = 1.$$

   2. Now, find the orthogonal projection of the second vector $v_2 = x$ onto u_1:

   $$proj_{u_1} v_2 = \frac{\langle x,1\rangle}{\langle 1,1\rangle} u_1 = \left(\int_0^1 x \cdot 1dx\right) 1 = \frac{1}{2}.$$

   Subtract this projection from v_2 to obtain the orthogonalized vector:

$$v_2' = v_2 - proj_{u_1} v_2 = x - \frac{1}{2}.$$

Normalize v_2' to get the second orthonormal basis vector:

$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 (x - \frac{1}{2})^2 dx}} = \sqrt{3}(2x - 1).$$

3. Finally, find the orthogonal projection of the third vector $v_3 = x^2$ onto u_1 and u_2:

$$proj_{u_1} v_3 = \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} u_1 = \left(\int_0^1 x^2 \cdot 1 dx \right) 1 = \frac{1}{3},$$

$$proj_{u_2} v_3 = \frac{\langle x^2, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = \int_0^1 x^2 \cdot \sqrt{3}(2x - 1) dx u_2 = \frac{\sqrt{3}}{6} u_2.$$

Subtract these projections from v_3 to obtain the orthogonalized vector:

$$v_3' = v_3 - proj_{u_1} v_3 - proj_{u_2} v_3 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}.$$

Normalize v_3' to get the third orthonormal basis vector:

$$u_3 = \frac{v_3'}{\|v_3'\|} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2 dx}} = \sqrt{180}(x^2 - x + \frac{1}{6}) = \sqrt{5}(6x^2 - 6x + 1).$$

(ii) Find the matrix representing differentiation on the basis $B = \{1, x, x^2\}$ and then on the basis obtained in (i). Check that both of these are upper-triangular. This is an example of Lemma 45 in the Lecture Notes.

Solution. *On the basis B_V the matrix representing $D : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ given by $D(p(x)) = p'(x)$ is:*

$$\mathcal{M}(D, B_V) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

On the orthonormal basis $B_V' = \{1, \sqrt{3}(2x - 1), \sqrt{5}(6x^2 - 6x + 1)\}$ just obtained we have:

$$\mathcal{M}(D, B_V') = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}$$

(iii) Consider the linear functional:

$$\begin{aligned} \lambda : \mathcal{P}_2(\mathbb{R}) &\to \mathbb{R} \\ p &\mapsto \int_0^1 \cos(\pi x) p(x) dx. \end{aligned}$$

Determine $q \in \mathcal{P}_2(\mathbb{R})$ such that $\lambda(-) = \langle -, q \rangle$, which exists by Riesz representation theorem.

Solution. *We apply formula (6.43) from the textbook. Thus,*

$$q = \lambda(u_1)u_1 + \lambda(u_2)u_2 + \lambda(u_3)u_3.$$

We compute:

$$\lambda(u_1) = \int_0^1 \cos(\pi x)dx = 0,\ \lambda(u_2) = \int_0^1 \sqrt{3}(2x-1)\cos(\pi x)dx = -\frac{4\sqrt{3}}{\pi^2}$$

and

$$\lambda(u_3) = \int_0^1 (\sqrt{5}(6x^2 - 6x + 1))\cos(\pi x)dx = 0.$$

Thus, $\lambda(p) = \left\langle p, -\frac{4}{\pi^2}(2x-1)\right\rangle$ *for every* $p \in \mathcal{P}_2(\mathbb{R})$.

4) (Gershgorin Disk Theorem) Let $T : V \to V$ be a linear operator and $B_V = \{v_1, \dots, v_n\}$ be a basis of V. Let $(a_{i,j})_{1\leq i,j\leq n} = \mathcal{M}(T, B_V)$ denote the matrix representing T in the basis B_V. For each $i \in \{1, \dots, n\}$ we define the *ith Gershgorin disk* to be:

$$D_i := \{z \in \mathbb{F} \mid |z - a_{i,i}| \leq \sum_{j\neq i} |a_{i,j}|\}.$$

(i) Prove that each eigenvalue λ of T belongs to D_i for some $i \in \{1, \dots, n\}$.

Solution. *This is Theorem 5.67 in the textbook.*

(ii) Assume that $\sum_{j\neq i} |a_{i,j}| < |a_{i,i}|$ for every $i \in \{1, \dots, n\}$. Prove that T is invertible. Can you give an example of a matrix such that this results allows you to deduce that it is invertible?

Solution. *Notice that (i) implies that each eigenvalue* λ *of* T *belongs to*

$$|\lambda| \in (a_{i,i} - \sum_{j\neq i} |a_{i,j}|, a_{i,i} + \sum_{j\neq i} |a_{i,j}|).$$

The condition given guarantees that $a_{i,i} - \sum_{j\neq i} |a_{i,j}| > 0$. *Thus, all eigenvalues of* T *are non-zero, so* T *is invertible.*

(iii) Let $D_i^{\text{col.}} := \{z \in \mathbb{F} \mid |z - a_{i,i}| \leq \sum_{j\neq i} |a_{j,i}|\}$, i.e. one change the definition of disks to compare the value of the diagonal entries with the other values in its column. Check that the same statement as in (i) holds for $D_i^{\text{col.}}$.

Solution. *One can simply follow the argument in Theorem 5.67 in the textbook for the transpose matrix.*