Math 2102: Worksheet 5

1) Let $T \in \mathcal{L}(V)$ and assume we are given $U_1, \ldots, U_n \subseteq V$ subspaces which are invariant under T.

   (i) Prove that $U_1 + \cdots + U_n$ is invariant under T.

   (ii) Prove that $U_1 \cap \cdots \cap U_n$ is invariant under T.

2) Prove or give a counter-example. Let $U \subseteq V$ be a subspace that is invariant under every operator $T \in \mathcal{L}(U)$, then $U = \{0\}$ or $U = V$.

3) (i) Consider $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ given by $T(p)(x) = p'(x)$. Find all eigenvalues and eigenvectors of T.

   (ii) Same as (i) but consider $T : \mathcal{P}_4(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$.

4) Let $T \in \mathcal{L}(V)$ and consider $S \in \mathcal{L}(V)$ an invertible operator.

   (i) Prove that T and STS^{-1} have the same eigenvalues.

   (ii) What is the relation between the eigenvectors of T and those of STS^{-1}?

5) Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$ an operator. Prove that $p_T = p_{T^*}$, where p_T is the minimal polynomial of T and p_{T^*} is the minimal polynomial of $T^* : V^* \to V^*$ the dual of T.

6) Let V be a finite-dimensional complex vector space and $T : V \to V$ and operator that only has eigenvalues 5 and 6. Prove that $(T - 5\,\mathrm{Id}_V)^{\dim V - 1}(T - 6\,\mathrm{Id}_V)^{\dim V - 1} = 0$.

7) Let V be a vector space of dimension d. Suppose that $T \in \mathcal{L}(V)$ is such that every subspace of dimension $k \in \{1, \ldots, d-1\}$ is invariant under T. Prove that T is a scalar multiple of the identity.

8) Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$ an operator and $p \in \mathbb{F}[x]$ its minimal polynomial.

   (i) Given $\lambda \in \mathbb{F}$ prove that $T - \lambda\,\mathrm{Id}_V$ has minimal polynomial $q(x) = p(x + \lambda)$.

   (ii) Given $\lambda \in \mathbb{F}\backslash\{0\}$ prove that λT has minimal polynomial $q(x) = \lambda^{\deg p} p\left(\frac{x}{\lambda}\right)$.

   (iii) Consider the subspace $E := \{q(T) \mid q \in \mathbb{F}[x]\} \subseteq V$. Prove that $\dim E = \deg p$.

   (iv) Prove that $\deg p \leq 1 + \dim \operatorname{range} T$.