Math 2102: Worksheet 5
Solutions

1) Let $T \in \mathcal{L}(V)$ and assume we are given $U_1, \dots, U_n \subseteq V$ subspaces which are invariant under T.

   (i) Prove that $U_1 + \cdots + U_n$ is invariant under T.

   Solution. *Let $v \in U_1 + \cdots + U_n$ then there exist $u_i \in U_i$ for $i \in \{1, \dots, n\}$ such that $v = \sum_{i=1}^{n} u_i$ then we have*

$$T(v) = T(\sum_{i=1}^{n} u_i) = \sum_{i=1}^{n} Tu_i \in U_1 + \cdots + U_n,$$

   since each $Tu_i \in U_i$.

   (ii) Prove that $U_1 \cap \cdots \cap U_n$ is invariant under T.

   Solution. *Let $v \in U_1 \cap \cdots \cap U_n$ then $v \in U_i$ for all $i \in \{1, \dots, n\}$, so $T(v) \in U_i$ for all $i \in \{1, \dots, n\}$. Thus, $T(v) \in U_1 \cap \cdots \cap U_n$.*

2) Prove or give a counter-example. Let $U \subseteq V$ be a subspace that is invariant under every operator $T \in \mathcal{L}(U)$, then $U = \{0\}$ or $U = V$.

   Solution. *To prove this statement, we will assume that $U \neq \{0\}$ and show that $U = V$.*

   Let $u \in U$ be a non-zero vector. Extend this to a basis $\{u, v_2, \dots, v_n\}$ of V. Then there are linear maps $\mathcal{L}_i$ that send u to v_i for $i \in \{2, \dots, n\}$. If U is invariant under $\mathcal{L}_i$ this implies that $\{u, v_2, \dots, v_n\} \subset U$, which gives that $V = \text{Span}\{u, v_2, \dots, v_n\} \subseteq U$, thus $U = V$.

3) (i) Consider $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ given by $T(p)(x) = p'(x)$. Find all eigenvalues and eigenvectors of T.

   Solution. *Let $p(x)$ be a non-zero polynomial such that $p'(x) = \lambda p(x)$ for some $\lambda \in \mathbb{F}$. Notice that since $\deg p'(x) < \deg(\lambda p(x))$, this implies that $p'(x) = 0$ and $\lambda p(x) = 0$. Since we want a non-zero p, this implies that $\lambda = 0$ and we have $p(x) = a$ for some $a \in \mathbb{F}$, the constant polynomial as an eigenvector. These are all the eigenvalues and eigenvectors.*

   (ii) Same as (i) but consider $T : \mathcal{P}_4(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$.

   Solution. *One has the same answer as in (i).*

4) Let $T \in \mathcal{L}(V)$ and consider $S \in \mathcal{L}(V)$ an invertible operator.

   (i) Prove that T and STS^{-1} have the same eigenvalues.

   Solution. *Let λ be an eigenvalue of T, that is there exists a non-zero vector $v \in V$ such that $T(v) = \lambda v$. Consider $w := S(v) \in V$, then we have:*

$$STS^{-1}(w) = ST(v) = S(\lambda v) = \lambda w.$$

   Since w is non-zero, otherwise S would not be injective, we obtain that $T(v)$ is an eigenvector of STS^{-1} with eigenvalue λ.
   Let λ be an eigenvalue of STS^{-1}, that is there exists a non-zero vector $v \in V$ such that $STS^{-1}(v) = \lambda v$. Consider $w := S^{-1}(v) \in V$ then we have

$$T(w) = TS^{-1}(v) = S^{-1}(\lambda v) = \lambda w.$$

   Thus, w is an eigenvector of T with eigenvalue λ.

(ii) What is the relation between the eigenvectors of T and those of STS^{-1}?

Solution. *The are in bijection. Let $E_T = \{v_1, \ldots, v_n\} \subset V$ be the subset of eigenvectors of T and $E_{STS^{-1}} = \{w_1, \ldots, w_n\}$ be the subset of eigenvectors of STS^{-1}. Then (i) just showed that:*

$$S : E_T \xrightarrow{\sim} E_{STS^{-1}}$$

is a bijection.

5) Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$ an operator. Prove that $p_T = p_{T^\vee}$, where p_T is the minimal polynomial of T and $p_{T^\vee}$ is the minimal polynomial of $T^\vee : V^\vee \to V^\vee$ the dual of T.

Solution. *We will repeatedly use that given $v \in V$ if $\alpha(v) = 0$ for all $\alpha \in V^\vee$ then $v = 0$. Similarly, given $\beta \in V^\vee$ if $\beta(u) = 0$ for all $u \in V$ then $\beta = 0$. This holds because V is finite-dimensional and can be easily proved by considering the dual basis.*

Let p_T be the minimal polynomial of T, and $p_{T^\vee}$ be the minimal polynomial of $T^\vee$. We want to show that $p_T = p_{T^\vee}$.

First, let's show that $p_{T^\vee}(T) = 0$. We need to check that for every $v \in V$ we have $p_{T^\vee}(T)(v) = 0$. Notice that for any $\alpha \in V^\vee$, we have

$$\alpha(p_{T^\vee}(T)(v)) = p_{T^\vee}(T^\vee)(\alpha(v)) = p_{T^\vee}(T^\vee)(\alpha)(v) = 0.$$

By the first paragraph we have that $p_{T^\vee}(T)(v) = 0$.

Now, let's show that $p_T(T^\vee) = 0$. For any $\alpha \in V^\vee$ and $v \in V$ we have

$$(p_T(T^\vee))(\alpha)(v) = \alpha(p_T(T(v))) = \alpha(p_T(T)(v)) = 0.$$

Again by the first paragraph we get that $p_T(T^\vee)(\alpha) = 0$ for any $\alpha \in V^\vee$.

Since $p_{T^\vee}(T) = 0$ and $p_T(T^\vee) = 0$ we obtain that $p_{T^\vee}$ divides p_T and that p_T divides $p_{T^\vee}$ as they are both monic this implies that $p_T = p_{T^\vee}$.

6) Let V be a finite-dimensional complex vector space and $T : V \to V$ and operator that only has eigenvalues 5 and 6. Prove that $(T - 5\,\mathrm{Id}_V)^{\dim V - 1}(T - 6\,\mathrm{Id}_V)^{\dim V - 1} = 0$.

Solution. *Since V is a complex vector space, by Lemma 32 in the Lecture Notes its minimal polynomial is*

$$p_T(z) = \prod_{i=1}^{n}(z - \lambda_i)$$

where λ_i are the eigenvalues of T. Thus, $p_T(z) = (z-5)^d(z-6)^e$ for some $d, e \geq 1$. Moreover, $\deg p_T \leq \dim V$, by Proposition 5 in the Lecture Notes. Thus, $d + e \leq \dim V$, which implies that $d, e \leq \dim V - 1$. So $q(z) = (z-5)^{\dim V - 1}(z-6)^{\dim V - 1}$ is a multiple of $p_T(z)$ and by Lemma 33 we have that $q(T) = 0$.

7) Let V be a vector space of dimension d. Suppose that $T \in \mathcal{L}(V)$ is such that every subspace of dimension $k \in \{1, \ldots, d-1\}$ is invariant under T. Prove that T is a scalar multiple of the identity.

Solution. *Suppose that $T \in \mathcal{L}(V)$ is such that every subspace of dimension $k \in \{1, \ldots, d-1\}$ is invariant under T. We will prove that T is a scalar multiple of the identity by showing that the action of T on any vector $v \in V$ is a scalar multiple of v.*

Let $v \in V$ be an arbitrary non-zero vector. Consider the one-dimensional subspace $U = \operatorname{span}\{v\}$. By assumption, U is invariant under T, which means that $T(U) \subseteq U$. Since U is one-dimensional, we have $T(v) = \lambda_v v$ for some scalar λ_v.

Now, let $w \in V$ be another non-zero vector linearly independent of v. Consider the two-dimensional subspace $W = \operatorname{span}\{v, w\}$. By assumption, W is invariant under T, which means that $T(W) \subseteq W$. Since W is two-dimensional, we have $T(w) = \lambda_w w$ for some scalar λ_w.

Let $x = av + bw$ be an arbitrary vector in V, where $a, b \in \mathbb{F}$. Then,

$$T(x) = T(av + bw) = aT(v) + bT(w) = a(\lambda_v v) + b(\lambda_w w).$$

Now, since v and w are linearly independent, the only way for $T(x)$ to be a scalar multiple of x is if $\lambda_v = \lambda_w = \lambda$ for some scalar λ. In this case, we have

$$T(x) = a(\lambda v) + b(\lambda w) = \lambda(av + bw) = \lambda x.$$

Thus, the action of T on any vector $x \in V$ is a scalar multiple of x. This proves that T is a scalar multiple of the identity.

8) Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$ an operator and $p \in \mathbb{F}[x]$ its minimal polynomial.

(i) Given $\lambda \in \mathbb{F}$ prove that $T - \lambda \operatorname{Id}_V$ has minimal polynomial $q(x) = p(x + \lambda)$.

Solution. *Let r be the minimal polynomial of $T - \lambda \operatorname{Id}_V$. Notice that $q(T - \lambda \operatorname{Id}_V) = p(T - \lambda \operatorname{Id}_V + \lambda \operatorname{Id}_V) = p(T) = 0$. So r divides q. Similarly, we have that $s(x) := r(x - \lambda)$ satisfies $s(T) = 0$.*

Now, we notice that q is also monic and of degree equal to p. Assume that $r \neq q$, then there exists s monic polynomial of degree less than p such that $s(T) = 0$. Thus, $r = q$ and we are done.

(ii) Given $\lambda \in \mathbb{F} \setminus \{0\}$ prove that λT has minimal polynomial $q(x) = \lambda^{\deg p} p\left(\frac{x}{\lambda}\right)$.

Solution. *Notice that $q(\lambda T) = \lambda^{\deg p} p(T) = 0$. Thus, p divides q. Notice that $\deg q = \deg p$, so $q = ap$ for some constant $a \in \mathbb{F}$. We notice that considering the coefficient of $x^{\deg p}$ that $a = 1$; thus we are done.*

(iii) Consider the subspace $E := \{q(T) \mid q \in \mathbb{F}[x]\} \subseteq V$. Prove that $\dim E = \deg p$.

Solution. *Claim :*

$$E = \{q(T) \mid q \in \mathbb{F}[x] \text{ and } deg\ q < \deg p_T\}.$$

Proof: Let $q \in \mathbb{F}[x]$ with deg $q \geq \deg p_T$. Then by the division algorithm, there exists $s, r \in \mathbb{F}[x]$ (where deg $r <$ deg p_T) such that $q = sp_T + r$. Then $q(T) = s(T)p_T(T) + r(T) = r(T)$ since p_T is the minimal polynomial of T. So the claim is proved.

Now consider the subset $S = \{\operatorname{Id}_V, T, \ldots, T^{\deg p_T - 1}\} \subset E$. We claim that S is linearly independent. Indeed, if that were not the case we could find $q(T) = 0$ monic such that $\deg q < \deg p_T$, which is a contradiction. Thus, $\dim E = |S| = \deg p_T$.

(iv) Prove that $\deg p \leq 1 + \dim \operatorname{range} T$.

Solution. *We proceed by induction on $\dim \operatorname{range} T$. If $\dim \operatorname{range} T = 0$, then $T = 0$ and $p_T(z) = z$ so $\deg p_T = 1$. Assume the result holds for every T such that $\dim \operatorname{range} T < k$.*

Notice that if $\operatorname{range} T = V$, then the result follows from the bound $\deg p \leq \dim V$. So we assume that T is not surjective. Consider $T' : \operatorname{range} T \to \operatorname{range} T$ the restriction of T to range T. *Let* $p_{T'}$ *be the minimal polynomial of* T'. *Since* $p_T(T') = 0$, p_T *is a multiple of* $p_{T'}$. *Now we notice that* $q(x) := p_{T'}(x)x$ *is such that* $q(T) = 0$*; so* q *is a multiple of* p_T. *In particular, we obtain*

$$\deg p_T \leq \deg q = \deg p_{T'} + 1 \leq \dim \operatorname{range} T' + 1 \leq \dim \operatorname{range} T + 1.$$