Math 2102: Worksheet 4

1) Suppose that x, y are vectors in a vector space V and let $U, W \subseteq V$ be two subspaces. Assume that $U + x = W + y$, prove that $U = W$.

2) Let $U \subseteq V$ be a subspace and assume that V/U is finite-dimensional. Prove that $V \simeq U \times V/U$.

3) Let $T \in \mathcal{L}(V, W)$ and consider $U \subseteq V$. Let $\pi : V \to V/U$ denote the quotient map. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $S \circ \pi = T$ if and only if $U \subseteq \operatorname{null} T$.

4) Let $\alpha, \beta \in V^*$. Prove that $\operatorname{null} \alpha \subseteq \operatorname{null} \beta$ if and only if $\beta = c\alpha$ for some $c \in \mathbb{F}$.

5) Let W be a finite-dimensional vector space and consider $T \in \mathcal{L}(V, W)$.

   (i) Prove that $T = 0$ if and only if $T^* = 0$.

   (ii) Is the same true if W is not finite-dimensional?

6) Let V be a finite-dimensional vector space. Consider $\lambda_1, \dots, \lambda_m \in V^*$ a collection of linearly independent (linear) functionals. Prove that

$$\dim((\operatorname{null} \lambda_1) \cap \cdots \cap (\operatorname{null} \lambda_m)) = \dim V - m.$$

7) Let $m \leq n$ be two positive integers. Consider $\alpha_1, \dots, \alpha_m \in \mathbb{F}$. Prove that there exists a polynomial $p \in \mathbb{F}[x]$ of degree n such that $p(\alpha_i) = 0$ for $1 \leq i \leq m$ and p has no other zeroes.

8) Let $m \geq 1$ be an integer and consider $z_1, \dots, z_m \in \mathbb{F}$ distinct elements and $w_1, \dots, w_m \in \mathbb{F}$ (not necessarily distinct). Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that $p(z_i) = w_i$ for $1 \leq i \leq m$.

9) Let $p \in \mathbb{C}[x]$ be a polynomial with complex coefficients. Define $q : \mathbb{C} \to \mathbb{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$, where $\overline{p(\bar{z})}$ is the polynomial obtained by conjugating all of the complex coefficients of $p(\bar{z})$. Prove that q is a polynomial with real coefficients.

10) Let $p \in \mathbb{F}[x]$ be a non-zero polynomial. Consider $U := \{pq \mid q \in \mathbb{F}[x]\} \subseteq \mathbb{F}[x]$.

    (i) Show that $\dim \mathbb{F}[x]/U = \deg p$.

    (ii) Find a basis of $\mathbb{F}[x]/U$.