Math 2102: Worksheet 4
Solutions

1) Suppose that x, y are vectors in a vector space V and let $U, W \subseteq V$ be two subspaces. Assume that $U + x = W + y$, prove that $U = W$.

Solution. *First we notice that* $x - y \in U \cap W$. *Indeed, we have*

$$0 + (y - x) \in W \ \Rightarrow x - y \in W,$$

since $0 \in U$ *and similarly*

$$0 + (x - y) \in u \ \Rightarrow x - y \in U.$$

Now, let $u \in U$ *we have* $u + (y - x) \in W$, *since* $x - y \in W$ *we have* $u + (y - x) + (x - y) \in W$, *i.e.* $u \in W$. *Similarly, we prove that* $W \subseteq U$.

2) Let $U \subseteq V$ be a subspace and assume that V/U is finite-dimensional. Prove that $V \simeq U \times V/U$.

Solution. *Let* $\{v_1 + U, \ldots, v_n + U\}$ *be a basis of* V/U. *Then* $\{v_1, \ldots, v_n\}$ *is a linearly independent subset of* V. *Consider the map* $\varphi : U \times V/U \to V$ *defined as:*

$$\varphi(u, x) := u + \sum_{i=1}^{n} a_i v_i,$$

where $x = \sum_{i=1}^{n} a_i(v_i + U)$ *is the expansion of* x *in* V/U. *We claim that* φ *is an isomorphism.*
First we prove that φ *is linear. Let* $(u, x), (u', x') \in U \times V/U$ *and* $b \in \mathbb{F}$ *we have:*

$$\begin{aligned}\varphi(u + bu', x + bx') &= u + au' + \sum_{i=1}^{n}(a_i + ba_i')v_i \\ &= u + \sum_{i=1}^{n} a_i v_i + au' \sum_{i=1}^{n} ba_i' v_i \\ &= \varphi(u, x) + b\varphi(u', x'),\end{aligned}$$

where $x' = \sum_{i=1}^{n} a_i'(v_i + U)$ *and* x *is written as before.*
Now we check that φ *is injective. Assume that*

$$u + \sum_{i=1}^{n} a_i v_i = u' + u + \sum_{i=1}^{n} a_i' v_i$$

for some a_i, a_i'*'s. Since* $v_i \neq U$ *by definition, otherwise* $v_i + U = 0 + U$, *so* $v_i + U$ *would not be part of a basis of* V/U, *we have that* $u - u'$ *and* $\sum_{i=1}^{n}(a_i - a_i')v_i = 0$, *which imply that* $a_i = a_i'$ *for all* $i \in \{1, \ldots, n\}$.
Finally, we check that φ *is surjective. Let* $v \in V$, *then consider* $\pi(v) \in V/U$ *we know that* $\pi(v) = \sum_{i=1}^{n} a_i(v_i + U)$, *i.e.*

$$v - \sum_{i=1}^{n} a_i v_i \in U \ \Rightarrow \ v = u + \sum_{i=1}^{n} a_i v_i.$$

Thus, $\varphi((u, \sum_{i=1}^{n} a_i(v_i + U))) = v$, *and we are done.*

3) Let $T \in \mathcal{L}(V, W)$ and consider $U \subseteq V$. Let $\pi : V \to V/U$ denote the quotient map. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $S \circ \pi = T$ if and only if $U \subseteq \operatorname{null} T$.

Solution. *Assume that such an S exists. Let $u \in U$, then $T(u) = S \circ \pi(u) = S(0) = 0$, thus $u \in \operatorname{null} T$.*

Now assume that $U \subseteq \operatorname{null} T$. We define: $S : V/U \to W$ by

$$S(v + U) := T(v).$$

We need to check that this is well-defined. Let $v + U = v' + U$ in V/U, then

$$S(v + U) = T(v) = T(v') - T(v' - v) = T(v'),$$

since $v' - v \in U$ so $T(v' - v) = 0$. Finally, we notice that $T(v) = S(v + U) = S \circ \pi(v)$, since $\pi(v) = v + U$.

4) Let $\alpha, \beta \in V^\vee$. Prove that $\operatorname{null} \alpha \subseteq \operatorname{null} \beta$ if and only if $\beta = c\alpha$ for some $c \in \mathbb{F}$.

Solution. *First assume that $\beta = c\alpha$ for some $c \in \mathbb{F}$. Then if $\alpha(v) = 0$ then $\beta(v) = 0$, so we get $\operatorname{null} \alpha \subseteq \operatorname{null} \beta$.*

Now assume that $\operatorname{null} \alpha \subseteq \operatorname{null} \beta$.

If $\alpha = 0$, then $V \subseteq \operatorname{null} \beta$, which implies that $\operatorname{null} \beta = V$, i.e. $\beta = 0$ and we are done.

So we assume that $\alpha \neq 0$. Let $v \in V$ such that $\alpha(v) \neq 0$. For every $u \in V$ notice that $\alpha(v)u - \alpha(u)v \in \operatorname{null} \alpha$, i.e. $\alpha(\alpha(v)u - \alpha(u)v) = 0$, so $\alpha(v)u - \alpha(u)v \in \operatorname{null} \beta$, which gives:

$$\beta(\alpha(v)u - \alpha(u)v) = 0 \quad \Rightarrow \quad \alpha(v)\beta(u) = \alpha(u)\beta(v),$$

that is $\beta(u) = \frac{\beta(v)}{\alpha(v)}\alpha(u)$ for every $u \in V$, i.e. $\beta = \frac{\beta(v)}{\alpha(v)}\alpha$. This finishes the proof.

5) Let W be a finite-dimensional vector space and consider $T \in \mathcal{L}(V, W)$.

(i) Prove that $T = 0$ if and only if $T^\vee = 0$.

Solution. *Assume that $T = 0$, consider $\varphi \in V^\vee$. Then we have*

$$T^\vee(\varphi)(v) = \varphi(T(v)) = \varphi(0).$$

So $T^\vee(\varphi) = 0$. Since φ was arbitrary we have that $T^\vee$.
Conversely, assume that $T^\vee = 0$. Let $v \in V$, then for every $\varphi \in V^\vee$ we have

$$T^\vee(\varphi)(v) = \varphi(T(v)) = 0.$$

Since W is finite-dimension consider a basis $\{e_1, \ldots, e_n\}$ of W which gives a basis $\{e_1^\vee, \ldots, e_n^\vee\}$ of $W^\vee$. Then we have $e_i^\vee(T(v)) = 0$, i.e. if $T(v) = \sum_{i=1}^n a_i e_i$ we have that $a_i = 0$ for all $i \in \{1, \ldots, n\}$. Since this happens for an arbitrary $v \in V$ we obtain that $T(v) = 0$.

(ii) (Extra) Is the same true if W is not finite-dimensional?

Solution. *This is not true in the infinite-dimensional case.*

6) Let V be a finite-dimensional vector space. Consider $\lambda_1, \ldots, \lambda_m \in V^\vee$ a collection of linearly independent (linear) functionals. Prove that

$$\dim((\operatorname{null} \lambda_1) \cap \cdots \cap (\operatorname{null} \lambda_m)) = \dim V - m.$$

Solution. *We proceed by induction on m.*

For $m = 1$ notice that $\{\lambda\}$ is linear independent implies that $\lambda \in V^\vee$ is non-zero. By the fundamental theorem of Linear Algebra we have

$$\dim V = \dim \operatorname{null} \lambda + \dim \operatorname{range} \lambda,$$

since $\lambda \neq 0$, there exists $v \in V$ such that $\lambda(v) \neq 0$ and we get that $\operatorname{range} \lambda = \mathbb{F}$, so $\dim \operatorname{range} \lambda = 1$. Thus $\dim \operatorname{null} \lambda = \dim V - 1$.

Let $V' := \operatorname{null} \lambda_1$ and $V' \oplus U = V$ be a decomposition of V. Let $\lambda_i' := \lambda_i|_{V'}$ for $i = 2, \ldots, m$. We claim that $\{\lambda_i'\}_{2 \leq i \leq n}$ is a linearly independent set. Assume by contradiction that there exists a non-trivial linear combination $\sum_{i=2}^m a_i \lambda_i'$. Then let $u \in U$ be any non-zero vector, in particular we have $\lambda_1(u) \neq 0$. We claim that

$$\frac{-\sum_{i=2}^m a_i \lambda_i(u)}{\lambda_1(u)} \lambda_1 + \sum_{i=2}^m a_i \lambda_i = 0 \tag{1}$$

is a non-trivial linear combination of $\{\lambda_1, \ldots, \lambda_m\}$. Indeed, we just need to check that (1) applied to any vector $v \in V$ vanishes. For $v \in V'$ this is clear by the choice of a_i. For any $w \in U$, since $\dim U = 1$ we have $w = bu$ for some $b \in \mathbb{F}$ and we obtain:

$$\frac{-\sum_{i=2}^m a_i \lambda_i(u)}{\lambda_1(u)} \lambda_1(bu) + \sum_{i=2}^m a_i \lambda_i(bu) = 0.$$

Now, by the inductive hypothesis we have that

$$\dim((\operatorname{null} \lambda_2') \cap \cdots (\operatorname{null} \lambda_m')) = \dim V' - (m - 1).$$

Since $(\operatorname{null} \lambda_1) \cap \cdots \cap (\operatorname{null} \lambda_m) = V' \cap (\operatorname{null} \lambda_2') \cap \cdots (\operatorname{null} \lambda_m') = (\operatorname{null} \lambda_2') \cap \cdots (\operatorname{null} \lambda_m')$ we get:

$$\dim(\operatorname{null} \lambda_1) \cap \cdots \cap (\operatorname{null} \lambda_m) = \dim V' - (m - 1) = \dim V - m,$$

where the last equality follows from the $m = 1$ case.

7) Let $m \leq n$ be two positive integers. Consider $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$. Prove that there exists a polynomial $p \in \mathbb{F}[x]$ of degree n such that $p(\alpha_i) = 0$ for $1 \leq i \leq m$ and p has no other zeroes.

   Solution. *The polynomial $p(x) := (x - \alpha_1)^{n-m+1} \prod_{i=2}^m (x - \alpha_i)$ does the trick.*

8) Let $m \geq 1$ be an integer and consider $z_1, \ldots, z_m \in \mathbb{F}$ distinct elements and $w_1, \ldots, w_m \in \mathbb{F}$ (not necessarily distinct). Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that $p(z_i) = w_i$ for $1 \leq i \leq m$.

   Solution. *Consider $p(z) := \sum_{j=1}^m \prod_{i \neq j} (z - z_i) \frac{w_j}{\prod_{i \neq j}(z_j - z_i)}$. This does the trick.*

9) Let $p \in \mathbb{C}[x]$ be a polynomial with complex coefficients. Define $q : \mathbb{C} \to \mathbb{C}$ by $q(z) = p(z)\overline{p(\bar{z})}$, where $\overline{p(\bar{z})}$ is the polynomial obtained by conjugating all of the complex coefficients of $p(\bar{z})$. Prove that q is a polynomial with real coefficients.

Solution. *Let* $p(z) = \sum_{i=0}^{n} a_i z^i$ *with* $a_i \in \mathbb{C}$*, then we have* $\overline{p(\overline{z})} = \sum_{i=0}^{n} \overline{a_i} z^i$*. Thus, we calculate:*

$$\begin{aligned} p(z)\overline{p(\overline{z})} &= (\sum_{i=0}^{n} a_i z^i)(\sum_{i=0}^{n} \overline{a_i} z^i) \\ &= \sum_{i=0}^{2n}(\sum_{j=0}^{i} a_j \overline{a_{i-j}})z^i. \end{aligned}$$

Thus, it is enough to check that for every $i \geq 0$ *we have* $\sum_{j=0}^{i} a_j \overline{a_{i-j}} \in \mathbb{R}$*. Notice that for* $i = 2k$ *for some* $k \geq 0$ *we have*

$$\sum_{j=0}^{i} a_j \overline{a_{i-j}} = |a_k|^2 + \sum_{j=0}^{k-1}(a_j \overline{a_{2k-j}} + \overline{a_j \overline{a_{2k-j}}}), \tag{2}$$

whereas for $i = 2k+1$ *for some* $k \geq 0$ *we have*

$$\sum_{j=0}^{i} a_j \overline{a_{i-j}} = \sum_{j=0}^{k} a_j (\overline{a_{2k+1-j}} + \overline{a_j \overline{a_{2k+1-j}}}). \tag{3}$$

Now, it is clear that every term in (2) and (3) is a real number.

10) Let $p \in \mathbb{F}[x]$ be a non-zero polynomial. Consider $U := \{pq \mid q \in \mathbb{F}[x]\} \subseteq \mathbb{F}[x]$.

  (i) Show that $\dim \mathbb{F}[x]/U = \deg p$.

  Solution. *By (ii) above we have a basis of* $\mathbb{F}[x]/U$ *with* d *elements, so* $\dim \mathbb{F}[x]/U = d$.

  (ii) Find a basis of $\mathbb{F}[x]/U$.

  Solution. *Let* $d = \deg p$*, we claim that* $\{1 + U, x + U, \ldots, x^{d-1} + U\}$ *is a basis of* $\mathbb{F}[x]/U$*. Indeed, it is clear that this set is linearly independent, otherwise we get*

$$x^i - x^j \in U \;\Rightarrow\; x^i - x^j = pq$$

for some q*, which gives a contradiction since* $\deg x^i - x^j \leq \max\{i, j\} \leq d-1$*, where* $\deg pq \geq d$*. Now let* $f \in \mathbb{F}[x]/U$*, then* $f = g + U$*, where* $g \in \mathbb{F}[x]$*. By Lemma 29 in the Lecture Notes, there exist* $a, b \in \mathbb{F}[x]$ *such that*

$$g = ap + b \quad \textit{and} \quad \deg b < d.$$

Thus, we have $g + U = b + U$*. Since* $\deg b < \deg p$*, we can find* $\beta_i \in \mathbb{F}$ *for* $i \in \{0, \ldots, d-1\}$ *such that* $b = \sum_{i=0}^{d-1} \beta_i x^i$*; thus*

$$g + U = \sum_{i=0}^{d-1} \beta_i (x^i + U).$$