Math 2102: Worksheet 3

Unless otherwise stated in the next exercises V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$.

1) Let $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.
2) Suppose that V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$.
3) Prove that there are bases B_V of V and B_W of W such that the matrix $\mathcal{M}(T, B_V, B_W)$ has all entries zero, except for the k entries in the diagonal where $1 \leq k \leq \dim \operatorname{range} M$.
4) Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 1.
5) Let B_V be a basis of V. Prove that there exists a basis B_W of W such that $\mathcal{M}(T, B_V, B_W)$ has all entries on the first column 0, except for possibly a 1 in the first row.
6) Let B_W be a basis of W. Prove that there exists a basis B_V of V such that $\mathcal{M}(T, B_V, B_W)$ has all entries on the first row 0, except for possibly a 1 in the first column.
7) Suppose that $T : V \to V$ is invertible prove that the following are equivalent:
   (i) T is invertible.
   (ii) $Tv_1, \ldots, Tv_n$ is a basis of V for every basis $v_1, \ldots, v_n$ of V.
   (iii) $Tv_1, \ldots, Tv_n$ is a basis of V for some basis $v_1, \ldots, v_n$ of V.
8) Let $S, T \in \mathcal{L}(V, W)$ prove that $\operatorname{range} T = \operatorname{range} S$ if and only if there exist an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.
9) Let $S, T \in \mathcal{L}(V, W)$ prove that $\operatorname{null} T = \operatorname{null} S$ if and only if there exist two invertible linear maps $D \in \mathcal{L}(V)$ and R such that $S = ETD$.
10) (i) For every $n \geq 1$, show that $V^n := V \times \cdots \times V$ (n times) and $\mathcal{L}(\mathbb{F}^n, V)$ are isomorphic.
   (ii) How many different isomorphisms are there in (i)?
   (iii) Let B_V be a basis of V and B be a basis of $\mathbb{F}^n$. Then we can associate to $v \in V$ two objects: a vector $\mathcal{M}(V, B_V)(v) \in \mathbb{F}^n$ and matrix $\mathcal{M}(L_v, B', B_V)$ where L_v is the linear operator you matched to v on (i). Is there any relation between these two objects? I.e. can you obtain one from the other?
11) Let $f : V \to W$ be a function between two vector spaces. Consider $\operatorname{graph} f := \{(v, w) \in V \times W \mid f(v) = w\} \subseteq V \times W$. Prove that f is linear if and only if $\operatorname{graph} f$ is a subspace of $V \times W$.
12) Let $U = \{(x_1, x_2, \ldots) \in \mathbb{F}^{\mathbb{N}} \mid \text{ only finitely many } x_i \neq 0\}$.
   (i) Show that U is a subspace of $\mathbb{F}^{\mathbb{N}}$.
   (ii) Prove that $\mathbb{F}^{\mathbb{N}}/U$ is finite-dimensional.