Math 2102: Worksheet 3
Solutions

Unless otherwise stated in the next exercises V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$.

1) Let $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.

   Solution. *Since differentiating lowers the degree of a polynomial by* 1*, we obtain a linear operator* $D : \mathcal{P}_{n+1}(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ *given by* $D(q) = 5q'' + 3q'$. *The Fundamental Theorem of Linear Algebra gives that*
   $$n + 2 = \dim \mathcal{P}_{n+1}(\mathbb{R}) = \dim \operatorname{null} D + \dim \operatorname{range} D.$$
   Notice that if $D(q) = 0$ *since* $\deg q'' < \deg q'$, *we have that* $q'' = 0$ *and* $q' = 0$, *which implies that* $q \in \mathcal{P}_0(\mathbb{R})$. *Thus,* $\dim \operatorname{null} D$ *and we obtain:*
   $$\dim \operatorname{range} D = n + 1 = \dim \mathcal{P}_n(\mathbb{R}).$$
   So D *is surjective, which implies that for any* $p \in \mathcal{P}_n(\mathbb{R})$ *we have a solution for the equation* $D(q) = p$.

   For the next questions assume that V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$.

2) Prove that there are bases B_V of V and B_W of W such that the matrix $\mathcal{M}(T, B_V, B_W)$ has all entries zero, except for the k entries in the diagonal where $1 \le k \le \dim \operatorname{range} T$.

   Solution. *Let* $B_{\operatorname{range} T} = \{f_1, \dots, f_l\}$ *be a basis of* $\operatorname{range} T$, *which we extend to* $B_W : \{f_1, \dots, f_l, f_{l+1}, \dots, f_m\}$ *a basis of* W. *For each* $f_i \in B_{\operatorname{range} T}$, *let* $v_i \in V$ *such that* $T(v_i) = f_i$. *Notice that* $\{v_1, \dots, v_l\}$ *is linearly independent. Indeed, if there is a linear combination* $\sum_{i=1}^{l} a_i v_i = 0$, *then* $T(\sum_{i=1}^{l} a_i v_i) = \sum_{i=1}^{l} a_i e_i = 0$, *which implies* $a_i = 0$ *for every* $i \in \{1, \dots, l\}$.

   Consider $B_V = \{v_1, \dots v_l, v_{l+1}, \dots, v_n\}$ *an extension of* $\{v_1, \dots, v_l\}$ *to a basis of* V. *Then we notice that*
   $$\mathcal{M}(T, B_V, B_W)_{i,j} = \begin{cases} 1 & \text{if } 1 \le j \le m, \text{ and } i = j; \\ 0 & \text{else.} \end{cases}$$

3) Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 1.

   Solution. *Assume all the entries of the matrix representing* T *are* 1. *Let* $B_V = \{v_1, \dots, v_n\}$. *We have that* $T(v_i)$ *is a multiple of* $T(v_j)$ *for any* $i = j$, *so* $\operatorname{Span}\{T(v_1)\} = \operatorname{Span}\{T(v_1), \dots, T(v_n)\}$. *Since* $\operatorname{range} T = \operatorname{Span}\{T(v_1), \dots, T(v_n)\}$ *we obtain* $\dim \operatorname{range} T = \dim \operatorname{Span}\{T(v_1)\} = 1$, *since this it* $w_1 + \cdots w_m$ *which is non-zero, where* $B_W = \{w_1, \dots, w_m\}$.

   Claim: *Let* $\{v_1, \dots, v_k\}$ *be a set of linearly independent vectors. Let* $w \in \operatorname{Span}\{v_2, \dots, v_k\}$ *then* $\{v_1 + w, v_2, \dots, v_k\}$ *is linear independent. Indeed, assume we have a linear combination:*
   $$a_1(v_1 + w) + \sum_{i=2}^{k} a_i v_i = 0 \;\Rightarrow\; a_1 v_1 = 0$$

which implies that $\sum_{i=2}^{k} a_i v_i = 0$*, thus* $a_i = 0$ *for* $i \geq 2$*. Similarly,* $\{v_1 + w, \ldots, v_{n-1} + w, v_n\}$ *where* $w \in \operatorname{Span}\{v_n\}$ *is linearly independent.*

Now assume that $\dim \operatorname{range} T = 1$*, by the Fundamental Theorem of Linear Algebra we have a basis* $\{v_1, \ldots, v_{n-1}\}$ *of* $\operatorname{null} T$*, which we extend to* $\{v_1, \ldots, v_n\}$ *a basis of* V*. Notice that* $B_V := \{v_1 + v_n, \ldots, v_{n-1} + v_n, v_n\}$ *is also a basis of* V *by the Claim above. Thus, by assumption we have*

$$T(v_i + v_n) = w$$

for some non-zero vector $w \in W$ *for all* $i \in \{1, \ldots, n-1\}$*. Let* $\{w, w_1, \ldots, w_{m-1}\}$ *be an extension to a basis of* W *and consider* $B_W := \{w - \sum_{i=1}^{m-1}, w_1 + w, \ldots, w_{m-1} + w\}$ *which is still a basis by the Claim. Then we claim that*

$$\mathcal{M}(T, B_V, B_W)_{i,j} = 1$$

for all $i \in \{1, \ldots, \dim W\}$ *and* $j \in \{1, \ldots, \dim V\}$*. Indeed, for every* $i \in \{1, \ldots, n-1\}$

$$T(v_i + v_n) = w = 1 \cdot (w - \sum_{i=1}^{m-1}) + \sum_{i=2}^{m} 1 \cdot w_i$$

and similarly $T(v_n) = w = 1 \cdot (w - \sum_{i=1}^{m-1}) + \sum_{i=2}^{m} 1 \cdot w_i$*.*

4) Let B_V be a basis of V. Prove that there exists a basis B_W of W such that $\mathcal{M}(T, B_V, B_W)$ has all entries on the first column 0, except for possibly a 1 in the first row.

   Solution. *Let* $B_V := \{e_1, e_2, \ldots, e_n\}$ *and consider* $w := T(e_1)$*. If* w *is non-zero we let* $B_W = \{w, f_2, \ldots, f_m\}$*. be an extension to a basis of* W*, else we let* B_W *be an arbitrary basis of* W*. It is clear that*

$$\mathcal{M}(T, B_V, B_W)_{i1} = \begin{cases} 1 & \text{if } w \neq 0 \\ 0 & \text{else} \end{cases}.$$

   This solves the exercise.

5) Let B_W be a basis of W. Prove that there exists a basis B_V of V such that $\mathcal{M}(T, B_V, B_W)$ has all entries on the first row 0, except for possibly a 1 in the first column.

   Solution. *Let* $B_W := \{f_1, f_2, \ldots, f_m\}$ *if* $f_1 \in \operatorname{range} T$ *we let* $e_1 \in V$ *such that* $T(e_1) = f_1$ *if* $f_1 \notin \operatorname{range} T$*, then we let* $e_1 \in V$ *be any non-zero vector. Consider* $B_V = \{e_1, \ldots, e_n\}$ *the extension to a basis of* V*. It is clear that*

$$\mathcal{M}(T, B_V, B_W)_{1i} = \begin{cases} 1 & \text{if } f_1 \in \operatorname{range} T \\ 0 & \text{else} \end{cases}.$$

6) Suppose that $T : V \to V$ is invertible prove that the following are equivalent:

   (1) T is invertible.

   (2) $Tv_1, \ldots, Tv_n$ is a basis of V for every basis $v_1, \ldots, v_n$ of V.

   (3) $Tv_1, \ldots, Tv_n$ is a basis of V for some basis $v_1, \ldots, v_n$ of V.

   Solution. *(1)* $\Rightarrow$ *(2): it is enough to check that* $\{Tv_1, \ldots, Tv_n\}$ *are linearly independent. Let* $\sum_{i=1}^{n} a_i T v_i = 0$ *be a linear combination, let* S *be the inverse of* T*, then we have:*

$$S(\sum_{i=1}^{n} a_i T v_i) = \sum_{i=1}^{n} a_i S(T v_i) = \sum_{i=1}^{n} a_i v_i = 0,$$

which implies that $a_i = 0$ *for every* $i \in \{1, \dots, n\}$.

(2) $\Rightarrow$ *(3): is obvious.*

(3) $\Rightarrow$ *(1): we define* $S : V \to V$ *by* $S(Tv_i) := v_i$. *We notice that for any* $v \in V$ *we have* $v = \sum_{i=1}^n a_i v_i$ *for some* $a_i \in \mathbb{F}$, *thus we get:*

$$ST(v) = S(\sum_{i=1}^n a_i T v_i) = \sum_{i=1}^n a_i S(Tv_i) = \sum_{i=1}^n a_i v_i = v.$$

Similarly, if we write $v = \sum_{i=1}^n b_i T v_i$ *for some* $b_i \in \mathbb{F}$, *we have:*

$$TS(v) = TS(\sum_{i=1}^n b_i T v_i) = T(\sum_{i=1}^n b_i S(Tv_i)) = T(\sum_{i=1}^n b_i v_i) = \sum_{i=1}^n b_i T v_i = v.$$

7) Let $S, T \in \mathcal{L}(V, W)$ prove that $\operatorname{range} T = \operatorname{range} S$ if and only if there exist an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.

Solution. *Assume that* $\operatorname{range} T = \operatorname{range} S$. *Let* $\{w_1, \dots, w_l\}$ *be a basis of* $\operatorname{range} T = \operatorname{range} S$. *For each* $j \in \{1, \dots, l\}$ *we let* $v_j \in V$ *and* $u_j \in V$ *such that*

$$T(v_j) = w_j = S(u_j).$$

Notice that $\{v_1, \dots, v_l\}$ *and* $\{u_1, \dots, u_l\}$ *are linearly independent subsets of* V. *We extend both of these to* $\{v_1, \dots, v_n\}$ *and* $\{u_1, \dots, u_n\}$ *bases of* V. *Notice that* $T(v_i) = 0$ *and* $S(u_i) = 0$ *for* $i \in \{l+1, \dots, n\}$, *since* $\operatorname{null} T \cap \operatorname{Span}\{v_{l+1}, \dots, v_n\} = \operatorname{null} T$ *and* $\operatorname{null} S \cap \operatorname{Span}\{u_{l+1}, \dots, u_n\} = \operatorname{null} S$ *by dimension reasons.*

Define:

$$E : V \to V, \qquad E(u_i) = v_i.$$

We notice that $F : V \to V$ *given by* $F(v_i) = u_i$ *is an inverse to* E. *Now we calculate:*

$$S(u_j) = \begin{cases} w_j & \text{if } j \in \{1, \dots, l\} \\ 0 & \text{else} \end{cases} = T(v_j) = T(E(u_j)).$$

As they agree on a basis, we have $S = TE$, *as desired.*

Conversely, assume that $S = TE$ *for some invertible* E. *Given* $w \in \operatorname{range} S$, *then* $W = S(u) = T(E(u))$, *so* $w \in \operatorname{range} S$. *Now assume that* $w \in \operatorname{range} T$, *then* $w = T(u)$ *for some* $u \in U$. *Let* $F : V \to V$ *be the inverse of* E. *Notice that* $S(F(u)) = T(E(F(u))) = T(u) = w$, *thus* $w \in \operatorname{range} S$.

8) Let $S, T \in \mathcal{L}(V, W)$ prove that $\dim \operatorname{null} T = \dim \operatorname{null} S$ if and only if there exist two invertible linear maps $D \in \mathcal{L}(V)$ and $E \in \mathcal{L}(W)$ such that $S = ETD$.

Solution. *Assume that* $\dim \operatorname{null} T = \dim \operatorname{null} S$. *Let* $\{e_1, \dots, e_k\}$ *be a basis of* $\operatorname{null} T$. *Extend this to* $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ *a basis of* V. *Let* $\{f_1, \dots, f_k\}$ *be a basis of* $\operatorname{null} T$. *Extend this to* $\{f_1, \dots, f_k, f_{k+1}, \dots, f_n\}$ *a basis of* V.

Let $B_T := \{Te_{k+1}, \dots, Te_n\} \subset W$ *and* $B_S := \{Sf_{k+1}, \dots, Sf_n\} \subset W$. *We claim that* B_T *is linearly independent subsets. Indeed, let* $\sum_{i=1}^{n-k} a_i T e_{k+i} = 0$ *be a linear combination, then* $T(\sum_{i=1}^{n-k} a_i e_{k+i}) = 0$, *which implies that* $\sum_{i=1}^{n-k} a_i e_{k+i} \in \operatorname{null} T$, *so* $\sum_{i=1}^{n-k} a_i e_{k+i} = \sum_{i=1}^{k} b_i e_i$ *for some* $b_i \in \mathbb{F}$, *which implies that* $a_i = 0$ *for all* $i \in \{1, \dots, n-k\}$. *Exactly the same argument gives that* B_S *is linearly independent.*

Consider $B_1 := \{Te_{k+1}, \dots, Te_n, w_1, \dots, w_k\}$ *be an extension to a basis of* W *and similarly let* $B_2 = \{Sf_{k+1}, \dots, Sf_n, w'_1, \dots, w'_k\}$ *be an extension to a basis of* W.

We now define $D : V \to V$ *by*

$$D(f_i) := e_i \text{ for } i \in \{1, \dots, n\}$$

and we define $E : W \to W$ *by*

$$E(Te_{k+i}) := Sf_{k+i} \text{ for } i \in \{1, \dots, n-k\} \text{ and } E(w_i) := w'_i \text{ for } i \in \{1, \dots, k\}.$$

Clearly, we have $S(f_i) = T(f_i)$ *for each* $i \in \{1, \dots, k\}$, *and*

$$S(f_{k+i}) = E(Te_{k+i}) = ETD(f_{k+i}) \text{ for } i \in \{1, \dots, n-k\}.$$

Thus, $S = ETD$.

Conversely, assume that there exist two invertible linear maps $D \in \mathcal{L}(V)$ *and* $E \in \mathcal{L}(W)$ *such that* $S = ETD$. *We claim that*

$$\dim \operatorname{null} ETD = \dim \operatorname{null} TD = \dim \operatorname{null} T. \tag{1}$$

Indeed, let $\{e_1, \dots, e_k\}$ *be a basis of* $\operatorname{null} T$. *We claim that* $\{D^{-1}e_1, \dots, D^{-1}e_k\}$ *is a basis of* $\operatorname{null} TD$. *Since* D^{-1} *is invertible it is clear that* $\{D^{-1}e_1, \dots, D^{-1}e_k\}$ *is linearly independent. Let* $v \in \operatorname{null} TD$, *then* $D(v) = \sum_{i=1}^k a_i e_i$ *for some* $a_i \in \mathbb{F}$, *thus* $v = \sum_{i=1}^k a_i D^{-1}(e_i)$. *Similarly, we can prove that if* $\{f_1, \dots, f_k\}$ *is a basis of* $\operatorname{null} TD$ *then* $\{Ef_1, \dots, Ef_k\}$ *is a basis of* $\operatorname{null} ETD$. *From (1) we obtain the claim.*

9) (i) For every $n \geq 1$, show that $V^n := V \times \cdots \times V$ (n times) and $\mathcal{L}(\mathbb{F}^n, V)$ are isomorphic.

Solution. *Let* $\{e_1, \dots, e_n\}$ *be the standard basis of* $\mathbb{F}^n$. *Consider the morphism*

$$V^{\times n} \to \mathcal{L}(\mathbb{F}^n, V)$$
$$v = (v_1, \dots, v_n) \mapsto T_v(e_i) := v_i.$$

Notice that $T : \mathbb{F}^n \to V$ *is defined by specifying what it does to the basis elements* $\{e_i\}_{1 \leq i \leq n}$ *of* $\mathbb{F}^n$.

By construction T *is linear so we need to show that it is bijective. Assume that* $T(v) = T(w)$ *then*

$$v_i = T(v)(e_i) = T(w)(e_i) = w_i \text{ for every } i \in \{1, \dots, n\},$$

so $v = w$.

Let $S \in \mathcal{L}(\mathbb{F}^n, V)$, *then we consider* $v := (S(e_1), \dots, S(e_n))$. *It is clear that*

$$S(e_i) = T_v(e_i) \text{ for every } i \in \{1, \dots, n\}$$

thus $T_v = S$. *This gives that* $v \mapsto T_v$ *is surjective.*

(ii) How many different isomorphisms are there in (i)?

Solution. *Many, we can permute the entries of* $V^{\times n}$, *e.g. consider*

$$V^{\times n} \to \mathcal{L}(\mathbb{F}^n, V)$$
$$v = (v_1, \dots, v_n) \mapsto T_v^\sigma(e_i) := v_{\sigma(i)},$$

where $\sigma \in S_n$. *We can also multiple each of the coordinates by a distinct non-zero scalar, e.g. for each* $\lambda := \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{F} \backslash \{0\}$ *we can consider:*

$$V^{\times n} \to \mathcal{L}(\mathbb{F}^n, V)$$
$$v = (v_1, \dots, v_n) \mapsto T_v^\lambda(e_i) := \lambda_i v_i.$$

Thus, there are at least $S_n \times (\mathbb{F} \backslash \{0\})^{\times n}$ *different isomorphisms.*

(iii) (Extra) Let B_V be a basis of V and B be a basis of $\mathbb{F}^n$. Then we can associate to $v \in V$ two objects: a vector $\mathcal{M}(V, B_V)(v) \in \mathbb{F}^n$ and matrix $\mathcal{M}(L_v, B, B_V)$ where L_v is the linear operator you matched to v on (i). Is there any relation between these two objects? I.e. can you obtain one from the other?

10) Let $f : V \to W$ be a function between two vector spaces. Consider $\operatorname{graph} f := \{(v, w) \in V \times W \mid f(v) = w\} \subseteq V \times W$. Prove that f is linear if and only if $\operatorname{graph} f$ is a subspace of $V \times W$.

Solution. *Assume that f is linear. Consider $(v, w), (v', w') \in \operatorname{graph} f$, i.e. $f(v) = w$ and $f(v') = w'$. Then $(v, w) + a(v', w') = (v + a'v', w + aw')$ and we have*

$$f(v + av') = f(v) + af(v') = w + aw' \;\Rightarrow\; (v + a'v', w + aw') \in \operatorname{graph} f.$$

Since $(0, 0) \in \operatorname{graph} f$, we see that $\operatorname{graph} f$ is a subspace.

Conversely, if $\operatorname{graph} f$ is a subspace, given $v, v' \in V$ we have $(v, f(v)) \in \operatorname{graph} f$ and $(av', af(v')) \in \operatorname{graph} f$ so $(v + av', f(v) + af(v'))) \in \operatorname{graph} f$ which gives that

$$f(v + av') = f(v) + af(v').$$

11) Let $U = \{(x_1, x_2, \ldots) \in \mathbb{F}^{\mathbb{N}} \mid \text{ only finitely many } x_i \neq 0\}$.

(i) Show that U is a subspace of $\mathbb{F}^{\mathbb{N}}$.

Solution. *Notice that $(0, 0, \ldots,) \in U$, so $U \neq \emptyset$.*
Let $x, y \in U$, then we have:

$$x + ay = (x_1 + ay_1, x_2 + ay_2, \ldots).$$

Assume that $x_i = 0$ for $i \notin S$, where $S \subset \mathbb{N}$ is finite and $y_i = 0$ for $i \notin T$, where $T \subset \mathbb{N}$ is also finite. Then it is clear that

$$x_i + ay_i = 0 \text{ for } i \notin T \cup S.$$

Since $T \cup S$ is finite, we have that $x + ay \in U$.

(ii) Prove that $\mathbb{F}^{\mathbb{N}}/U$ is infinite-dimensional.

Solution. *For any prime p, define $x_p = (x_{p,1}, x_{p,2}, \ldots)$ as follows:*

$$x_{p,i} = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{p} \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $0_{\mathbb{F}^N/U} \neq [x_p] = x_p + U \in \mathbb{F}^N/U$, since has infinitely many nonzero entries. Also for different prime numbers p, q, $x_p - x_q \notin U$ since $x_{p,q}(kp + 1) - x_{q,q}(kp + 1) = 1$ for all $k \in \mathbb{N}$.

Therefore the set $\{[x_p] \in \mathbb{F}^{\mathbb{N}}/U \,:\, \text{prime } p \in \mathbb{N}\}$ is an infinite subset of $\mathbb{F}^{\mathbb{N}}/U$ since there are infinitely many primes. Now if there exists $a_1, a_2, \cdots \in F$ such that

$$\sum_{i=1}^{\infty} a_i [x_p] = [0]$$

(where p is the i-th prime), then we have $\sum_{i=1}^{\infty} a_i x_p \in U$. So there exists $M \in \mathbb{N}$ such that for $i > M$, the i-th entry of the sum is zero. But now if we consider the $(p_M + 1)$-th entry

(note that $p_M > M$), since $(p_M + 1) - 1$ isn't divisible by any other p_i, we have $x_{p_i, p_M+1} = 0$ for all $i \neq M$. Thus the $(p_M + 1)$-th entry of $\sum_{i=1}^{\infty} a_i x_{p_i}$ is just a_M which is now zero. By the same argument $a_i = 0$ for all $i \geq M$. So now

$$\sum_{i=1}^{\infty} a_i x_p = \sum_{i=1}^{M-1} a_i x_p.$$

But again for any $1 \leq i \neq j \leq M - 1$, $p_i(Mp_1 \dots p_{i-1}p_{i+1} \dots p_{M-1} + 1)$ is not divisible by p_j but is divisible by p_i, so the $(p_i(Mp_1 \dots p_{i-1}p_{i+1} \dots p_{M-1} + 1) + 1)$-th entry of the sum is just a_i. But since $p_i(Mp_1 \dots p_{i-1}p_{i+1} \dots p_{M-1} + 1) + 1 > M$, we have $a_i = 0$. To conclude, $a_i = 0$ for all $i \in \mathbb{N}$.

Therefore the infinite subset $\{[x_p] \in \mathbb{F}^{\mathbb{N}}/U \ : \ \text{prime } p \in \mathbb{N}\}$ of $\mathbb{F}^{\mathbb{N}}/U$ is linearly independent which implies $\mathbb{F}^{\mathbb{N}}/U$ is infinite-dimensional.