Math 2102: Worksheet 2

1) Suppose that V is finite-dimensional and that $U, W \subset$ are subspaces such that $U + W = V$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

2) Let $U := \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 \mid 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}$.

   (i) Find a basis of U.

   (ii) Extend the basis of (i) to a basis of $\mathbb{C}^5$.

   (iii) Find a subspace $W \subset \mathbb{C}^5$ such that $V \oplus W = \mathbb{C}^5$.

3) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) \mid \int_{-1}^{1} p = 0\}$.

   (i) Find a basis of U.

   (ii) Extend the basis of (i) to a basis of $\mathcal{P}_4(\mathbb{R})$.

   (iii) Find a subspace $W \subset \mathcal{P}_4(\mathbb{R})$ such that $V \oplus W = \mathcal{P}_4(\mathbb{R})$.

4) Assume that $\{v_1, \ldots, v_m\}$ is a linearly independent subset of a vector space V. Let $w \in V$, prove that

$$\dim \operatorname{Span}(\{v_1 + w, \ldots, v_m + w\}) \geq m - 1.$$

5) Let V be a finite-dimensional vector space and $U \subset V$ a proper subspace, i.e. $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there are $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection is U.

6) Let V be a 1-dimensional vector space. Prove that every linear map $T : V \to V$ is given by multiplication by a scalar.

7) Can you come with examples of vector spaces V and W and functions $\varphi : V \to W$ such that φ satisfies either additivity or homogeneity, but *not* both.

8) Let $U \subset V$ be a subspace of a finite-dimensional vector space V. Let $\varphi : U \to W$ be a linear map, prove that there exists an extension $\overline{\varphi} : V \to W$ which is a linear map, i.e. for every $u \in U$ one has $\overline{\varphi}(u) = \varphi(u)$.

9) Given an example of a linear map T with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

10) Let $S, T \in \mathcal{L}(V)$ and assume that $\operatorname{range} S \subseteq \operatorname{null} T$. Prove that $(ST)^2 = 0$.

11) (a) Give an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$.

    (b) Prove that there exist no $T \in \mathcal{L}(\mathbb{R}^5)$ such that $\operatorname{range} T = \operatorname{null} T$.

12) Let $P \in \mathcal{L}(V)$ such that $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.