Math 2102: Worksheet 2
Solutions

1) Suppose that V is finite-dimensional and that $U, W \subset$ are subspaces such that $U + W = V$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

   Solution. *By definition of the sum of vector spaces,* $\operatorname{Span}(U \cup W) = V$. *The reduction theorem says we may reduce* $U \cup W$ *to a basis.*

2) Let $U := \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 \mid 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}$.

   (i) Find a basis of U.

   Solution. $\{(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)\}$ *is a basis of* U *as for any vectors in* U,

   $$(z_1, z_2, z_3, z_4, z_5) = z_1(1, 6, 0, 0, 0) + z_4(0, 0, -2, 1, 0) + z_5(0, 0, -3, 0, 1).$$

   (ii) Extend the basis of (i) to a basis of $\mathbb{C}^5$.

   Solution. $\{(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}$ *is a basis of* $\mathbb{C}^5$ *evidently.*

   (iii) Find a subspace $W \subset \mathbb{C}^5$ such that $V \oplus W = \mathbb{C}^5$.

   Solution. *By conclusion in (ii),* $W = \operatorname{Span}\{(0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}$.

3) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) \mid \int_{-1}^{1} p = 0\}$.

   (i) Find a basis of U.

   Solution. *Suppose* $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \in U$, *then*

   $$\int_{-1}^{1} p(x)\,dx = 2a_0 + \frac{2a_2}{3} + \frac{2a_4}{5} = 0.$$

   On the contrary, it is trivial that if $15a_0+5a_2+3a_4 = 0$ *then* $p \in U$, *so* $U \cong \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : 15a_0 + 5a_2 + 3a_4 = 0\}$ *by identifying* z^i *with* $(0, \ldots, 0, 1, 0, \ldots, 0)$, *where* 1 *is on the* i*-th slot. Hence, we may apply a similar method as 1).*

   $$\{x, x^3, -3 + x^2, -5 + x^4\}$$

   is a basis of U.

   (ii) Extend the basis of (i) to a basis of $\mathcal{P}_4(\mathbb{R})$.

   Solution.
   $$\{1, x, x^3, -3 + x^2, -5 + x^4\}$$

   is a basis of $\mathcal{P}_4(\mathbb{R})$.

   (iii) Find a subspace $W \subset \mathcal{P}_4(\mathbb{R})$ such that $V \oplus W = \mathcal{P}_4(\mathbb{R})$.

   Solution. $W = \mathbb{R}$ *will work.*

4) Assume that $\{v_1, \ldots, v_m\}$ is a linearly independent subset of a vector space V. Let $w \in V$, prove that

   $$\dim \operatorname{Span}(\{v_1 + w, \ldots, v_m + w\}) \geq m - 1.$$

Solution. *If* $w \notin \operatorname{Span}\{v_1, \dots, v_m\}$*, i.e.* $\{w, v_1, \dots, v_m\}$ *is linearly independent. Notice* $\{w, v_1 + w, \dots, v_m + w\}$ *generates the same space with the same amount of vectors; it is linearly independent, in particular, so is* $\{v_1 + w, \dots, w_m + w\}$*, and hence* $\dim \operatorname{Span}(\{v_1 + w, \dots, v_m + w\}) = m$*.*

Now suppose $w = \sum_{i=1}^{m} a_i v_i$*. If* $a_i = 0$ *for all* $i = 1, \dots, n$*, then the dimension is still* m*.*

Suppose $a_1 \neq 0$ *by rearranging the index if necessary. We may simply see that*

$$\operatorname{Span}\{w, v_2 + w, \dots, v_m + w\} = \operatorname{Span}\{w, v_2, \dots, v_m\} = \operatorname{Span}\{v_1, \dots, v_m\}.$$

[The idea is from the Gaussian eliminations.] Similarly, we may conclude that $\{w, v_2 + w, \dots, v_m + w\}$ *is linearly independent, in particular,* $\{v_2 + w, \dots, v_m + w\}$ *is linearly independent, and hence the desired inequality is proven.*

5) Let V be a finite-dimensional vector space and $U \subset V$ a proper subspace, i.e. $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there are $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection is U.

Solution. *Let* $\{u_1, \dots, u_m\}$ *be a basis of* U*, and we may extend it to a basis of* V*, denoted* $\{u_1, \dots, u_m, v_1, \dots, v_{n-m}\}$*. Since* $\{v_i : i = 1, \dots, n-m\}$ *is linearly independent,* $W_i := \operatorname{Span}\{u_1, \dots, u_m, v_1, \dots, v$ *is distinct for all* i*.*

We claim $\bigcap_{i=1}^{n-m} W_i = \operatorname{Span}\{u_i : i = 1, \dots, m\} = U$*. Indeed, since* $W_i \supseteq U$ *for all* i*, we have* $\bigcap_{i=1}^{n-m} W_i \supseteq U$*. Notice that* $u_j \in W_i$ *for all* $i = 1, \dots, n - m$ *and for all* $j = 1, \dots, m$*, thus* $\bigcap_{i=1}^{n-m} W_i \subseteq U$*, so we are done.*

6) Let V be a 1-dimensional vector space. Prove that every linear map $T : V \to V$ is given by multiplication by a scalar.

Solution. *Let* $\{v\}$ *be a basis of* V*. Since* $Tv \in V$*, we may find* $Tv = \lambda v$ *for some* $\lambda \in \mathbb{C}$*.*

For any $w \in V$*,* $w = av$ *for some* $a \in \mathbb{F}$*, then by the linearity,* $Tw = aTv = a\lambda v = \lambda w$*. Hence* $Tw = \lambda w$ *for all* $w \in V$*.*

7) Can you come with examples of vector spaces V and W and functions $\varphi : V \to W$ such that φ satisfies either additivity or homogeneity, but *not* both.

Solution. *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by* $(x, y) \mapsto \frac{x^3 + y^3}{x^2 + y^2}$*. It is evidently homogeneous but not additive as* $f(0, 1) + f(1, 0) = 2 \neq 1 = f(1, 1)$*.*

Consider $\mathbb{R}$ *as a vector space over* $\mathbb{Q}$*. By the axiom of choice, we may find a basis* $\{v_i : i \in I\}$ *of* $\mathbb{R}$*. If* $x, y \notin \operatorname{Span}_{\mathbb{Q}}\{v_i\}$*, then* $x + y \notin \operatorname{Span}_{\mathbb{Q}}\{v_i\}$ *by the linear independence. Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(qv_i) = q$ *and* $f = 0$ *otherwise. It is easy to see that* f *is additive but not (*$\mathbb{R}$*-)homogeneous.*

8) Let $U \subset V$ be a subspace of a finite-dimensional vector space V. Let $\varphi : U \to W$ be a linear map, prove that there exists an extension $\overline{\varphi} : V \to W$ which is a linear map, i.e. for every $u \in U$ one has $\overline{\varphi}(u) = \varphi(u)$.

Solution. *Let* $\{u_1, \dots, u_m\}$ *be a basis of* U*, then we may extend it to a basis of* V*, denoted* $\{u_1, \dots, u_m, v_1, \dots, v_n\}$*. Define* $\overline{\varphi}$ *by* $u_i \mapsto \varphi(u_i)$ *and* $v_j \mapsto 0$ *for all* $i = 1, \dots, m$ *and* $j = 1, \dots, n$*.*

To extend this result to infinite-dimensional situations one needs to develop more theory. One result in that direction is the Hahn-Banach Theorem (see here).

9) Given an example of a linear map T with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Solution. *Define T by $Te_1 = e_1$, $Te_2 = e_2$, and $Te_3 = Te_4 = Te_5 = 0$, where $\{e_i : i = 1, \ldots, 5\}$ is the standard basis of $\mathbb{R}^5$. One should check the range and kernel of T on their own.*

10) Let $S, T \in \mathcal{L}(V)$ and assume that $\operatorname{range} S \subseteq \operatorname{null} T$. Prove that $(ST)^2 = 0$.

Solution. *For any $v \in V$, $STv \in \operatorname{range} S \subset \operatorname{null} T$, and hence $STSTv = S(0) = 0$.*

11) (a) Give an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$.

Solution. *Define T by $Te_1 = e_3$, $Te_2 = e_4$, and $Te_3 = Te_4 = 0$, where $\{e_i : i = 1, \ldots, 4\}$ is the standard basis of $\mathbb{R}^4$. One should check the range and kernel of T on their own.*

(b) Prove that there exist no $T \in \mathcal{L}(\mathbb{R}^5)$ such that $\operatorname{range} T = \operatorname{null} T$.

Solution. *By the first isomorphism theorem of $\mathbb{R}$-vector spaces, $\operatorname{range} T + \operatorname{null} T = 5$. If $\operatorname{range} T = \operatorname{null} T$, then $\operatorname{range} T \notin \mathbb{N}$, which is absurd.*

12) Let $P \in \mathcal{L}(V)$ such that $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

Solution. *Suppose $v \in \operatorname{null} P \cap \operatorname{range} P$, then we may find $w \in V$ such that $Pw = v$, then $v = Pw = P^2 w = Pv = 0$. It is trivial that $0 \in \operatorname{null} P \cap \operatorname{range} P$, so $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.*

Suppose $\{u_1, \ldots, u_n\}$ is a basis of $\operatorname{null} P$ and $\{v_1, \ldots, v_r\}$ is a basis of $\operatorname{range} P$, where $n+r = \dim V$. Suppose $a_1 u_1 + \cdots + a_n u_n + b_1 v_1 + \cdots + b_r v_r = 0$, then

$$a_1 u_1 + \cdots + a_n u_n = -b_1 v_1 - \cdots - b_r v_r \in \operatorname{null} P \cap \operatorname{range} P = \{0\}.$$

Since $\{u_i\}$ is linearly independent, then $a_i = 0$ for all i. Similarly, $b_j = 0$ for all j. Therefore, we may conclude that $\{u_1, \ldots, u_n, v_1, \ldots, v_r\}$ is linearly independent, so $\dim(\operatorname{null} P \oplus \operatorname{range} P) = n + r = \dim V$. Since $\operatorname{null} P \oplus \operatorname{range} P \subset V$, we may conclude that $\operatorname{null} P \oplus \operatorname{range} P = V$.