Math 2102: Worksheet 1
Solutions

1) Exercise 6 from §1B.

   Solution. *Consider* $(2-1)\cdot\infty = 1\cdot\infty = \infty$, *however* $2\cdot\infty - 1\cdot\infty = \infty + (-\infty) = 0$. *So distributivity fails and* $V = \mathbb{R}\cup\{\infty\}\cup\{-\infty\}$ *is not a vector space.*

2) Let $U_1, U_2 \subseteq V$ be two subspaces. Prove that $U_1 \cup U_2$ is a subspace if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

   Solution. *Assume that* $U_1 \subseteq U_2$ *or* $U_2 \subseteq U_1$, *then we have* $U_1 \cup U_2 = U_i \subseteq V$ *for* $i \in \{1,2\}$, *thus* $U_1 \cup U_2$ *is a subspace.*

   Now assume that $U_1 \cup U_2$ *is a subspace and by contradiction assume that* $U_1 \not\subset U_2$ *and* $U_2 \not\subset U_1$. *Then there exists* $v_1 \in U_1$ *such that* $v_1 \notin U_2$ *and* $v_2 \in U_2$ *such that* $v_2 \notin U_1$. *But then we have* $v_1 + v_2 \in U_1 \cup U_2$, *which implies that either* $(v_1+v_2) - v_1 \in U_1$ *or* $(v_1+v_2) - v_2 \in U_2$, *i.e. either* $v_2 \in U_1$ *or* $v_1 \in U_2$, *which is a contradiction.*

3) Assume that $\mathbb{F}$ has not characteristic 2. Prove that the union of three subspaces is a subspace if and only if one of them contains the other two.

4) Exercises 16, 17 and 18 from §1C.

   Solution. *Notice that in class we argued that* $U_1 + U_2 = \operatorname{Span} U_1 \cup U_2$, *since they are both the smallest subspace of* V *containing* U_1 *and* U_2. *Thus, 16 and 17 follow from the fact that union of sets is commutative and associative.*

   For 18: firstly, we notice that $\{0\} + U = U$ *for any subspace* $U \subseteq V$, *since* $\{0\} \subseteq U$, *so* $\{0\} + U = \{0\} \cup U = U$. *Secondly, for* $U \subseteq V$ *if there exists* $W \subseteq V$ *such that* $U + W = \{0\}$, *then we have that* $U \subseteq \{0\}$, *which implies that* $U = \{0\}$.

5) Prove of give a counterexample: suppose that $V + U_1 = V + U_2$ then $U_1 = U_2$.

   Solution. *This is false. Consider* $V = \operatorname{Span}\{e_1, e_2\}$, $U_1 = \operatorname{Span}\{e_3\}$, *and* $U_2 = \operatorname{Span}\{e_1 + e_3\}$ *inside* $\mathbb{F}^3$. *Here* $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, *and* $e_3 = (0,0,1)$. *We have that* $V + U_1 = \mathbb{F}^3 = V + U_2$, *but clearly* $U_1 \neq U_2$.

6) Exercise 24 from §1C.

   Solution. *First we claim that* $V_e \oplus V_0 = \mathbb{R}^{\mathbb{R}}$. *Indeed, let* $f : \mathbb{R} \to \mathbb{R}$ *be any element of* $\mathbb{R}^{\mathbb{R}}$ *we define:*

   $$g_1 := f(x) + f(-x) \qquad \text{and} \qquad g_2 := f(x) - f(-x).$$

   Notice that $g_1 \in V_e$ *and* $g_2 \in V_0$. *Finally, we claim that* $V_e \cap V_0 = \{0\}$. *Indeed, if* $f(x) = f(-x) = -f(x)$ *for all* $x \in \mathbb{R}$, *this implies that* $f(x) = 0$ *for all* $x \in \mathbb{R}$.

7) Exercise 3 from §2A.

   Solution. *We prove the result by induction. It is clearly true for* $n = 1$.

   Inductive step. Assume the result holds for $n-1$. *Then,*

   $$\operatorname{Span}\{v_1, \ldots, v_{n-1}, v_n\} = \operatorname{Span}\{w_1, \ldots, w_{n-1}, v_n\}.$$

So it is enough to check that $\operatorname{Span}\{w_1, \ldots, w_{n-1}, v_n\} = \operatorname{Span}\{w_1, \ldots, w_{n-1}, w_n\}$. *Since* $v_n = w_n - w_{n-1}$ *we have* $\operatorname{Span}\{w_1, \ldots, w_{n-1}, v_n\} \subseteq \operatorname{Span}\{w_1, \ldots, w_{n-1}, w_n\}$. *Since* $w_n = v_n - w_{n-1}$ *we have* $\operatorname{Span}\{w_1, \ldots, w_{n-1}, w_n\} \subseteq \operatorname{Span}\{w_1, \ldots, w_{n-1}, v_n\}$. *This finishes the inductive set and finishes the proof.*

8) Exercise 7 from §2A.

Proof. (a) Assume that there exists $a, b \in \mathbb{R}$ such that $a(1+i)+b(1-i) = 0$. Then $(a+b)+(a-b)i = 0$ and since a complex number is zero if and only if both its real and imaginary part vanish, we have $a + b = 0$ and $a - b = 0$, whose only solution is $a = b = 0$.

(b) Assume that there exists $a, b \in \mathbb{C}$ such that $a(1 + i) + b(1 - i) = 0$. For instance we have $a = (1 - i)$ and $b = -(1 + i)$ is a non-zero solution. So $\{1 + i, 1 - i\}$ is *not* linearly independent over $\mathbb{C}$. □

9) Exercises 15 and 16 form §2A.

Solution. *15: Assume that there are* $\{v_1, \ldots, v_6\} \subseteq \mathcal{P}_4(\mathbb{F})$ *which are linearly independent. Notice that* $S = \{1, x, x^2, x^3, x^4\} \subset \mathcal{P}_4(\mathbb{F})$ *is a spanning set, i.e.* $\operatorname{Span} S = \mathcal{P}_4(\mathbb{F})$. *By (2.22) from the textbook we have* $6 = |\{v_1, \ldots, v_6\}|$ *is smaller than or equal to* $|S| = 5$. *This is a contradiction.*

16: Assume that there exist a list $T = \{v_1, \ldots, v_4\}$ *such that* $\operatorname{Span} T = \mathcal{P}_4(\mathbb{F})$. *Since* $S = \{1, x, x^2, x^3, x^4\}$ *is linearly independent, again by (2.22) we should have* $|S| \leq |T|$, *which is a contradiction.*