Math 2102 - Review problems
April 25, 2024

- The exercises marked with (*) are the ones that I expect most of you to know how to solve and you should expect similar level of question in the Final Exam.
- The exercises marked with (**) may be a bit more challenging or a bit of a digression. You should not get caught up in them if you don't know how to solve it.
- The exercises not marked I don't feel particularly strong in either direction.

I. Basic concepts + Fundamental Theorem of Linear Algebra

I.1) Let V be a finite-dimensional vector space and consider subspaces $U_1, U_2 \subseteq V$.

(i) (**) Assume that $\dim U_1 = \dim U_2$. Prove that there exist a subspace $W \subseteq V$ such that $U_1 \oplus W = U_2 \oplus W = V$.

(ii) Assume that $\dim U_i \leq m < \dim V$. Prove that there exist a subspace $W \subseteq V$ of dimension $\dim V - m$ such that $W \cap U_1 = W \cap U_2 = \{0\}$.

I.2) (*) Let $U \subseteq V$. Prove that there exist $T, S \in \mathcal{L}(V)$ such that $\operatorname{null} T = U$ and $\operatorname{range} S = U$.

I.3) (*) Let V be an arbitrary vector space.

(i) Consider $U \subset V$ a proper subspace, i.e. $U \neq V$. Prove or disprove U is not isomorphic to V.

(ii) Assume that we have subspaces $U_1, U_1, W_1, W_2 \subseteq V$ such that $U_1 \oplus W_1 = U_2 \oplus W_2 = V$ and that $U_1 \simeq U_2$, i.e. U_1 and U_2 are isomorphic. Prove or disprove W_1 and W_2 are isomorphic.

I.4) (*) Let $U \subset \mathbb{R}^8$ be a subspace of dimension 3. Let $T : \mathbb{R}^8 \to \mathbb{R}^5$ be a linear map such that $\operatorname{null} T = U$. Prove that T is surjective.

I.5) (*) Let $T : V \to W$ be a linear map and V a finite-dimensional vector space. Prove that there exist a subspace $U \subseteq V$ such that:

$$\operatorname{null} T \cap U = \{0\} \text{ and } \operatorname{range} T = \{T(u) \mid u \in U\}.$$

I.6) (*) Determine if the following are true or false and think of a brief explanation of why that is the case.

(i) Let V be a vector space and $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ a basis of V. Consider $U \subseteq V$ a subspace. Then

$$U = (U \cap \operatorname{Span}\{v_1, v_2\}) \oplus (U \cap \operatorname{Span}\{v_3\}) \oplus (U \cap \operatorname{Span}\{v_4, v_5, v_6\}).$$

(ii) Let $\{u_1, \ldots, u_n\}$ be a basis of V and $\{w_1, \ldots, w_m\}$ be a set of linearly independent vectors in V with $m \leq n$. There exist an unique $T : V \to V$ such that $T(v_i) = w_i$ for $i \leq m$ and $T(v_i) = 0$ for $i \geq m + 1$. Moreover, T is invertible when $n = m$.

(iii) Consider two linear maps $T, S : V \to W$. Then $\operatorname{null} T + \operatorname{null} S \subseteq \operatorname{null}(T + S)$.

(iv) Consider two linear maps $T, S : V \to W$. Then $\operatorname{null} T \cap \operatorname{null} S \subseteq \operatorname{null}(T + S)$.

(v) Consider two linear maps $T, S : V \to W$. Then

$$\dim \operatorname{range}(TS) \leq \min\{\dim \operatorname{range} T, \dim \operatorname{range} S\}.$$

(vi) Consider two linear maps $T, S : V \to W$. Then

$$\dim \operatorname{range}(T + S) = \dim \operatorname{range} T + \dim \operatorname{range} S.$$

II. **Matrix representations**

II.1) (*) Let $B_1 = \{v_1, \ldots, v_n\}$ and $B_2 = \{u_1, \ldots, u_n\}$ be two basis of V. Consider $T : V \to V$ defined by $Tv_i = u_k$. Prove that

$$\mathcal{M}(T, B_1) = \mathcal{M}(\mathrm{Id}_V, B_2, B_1).$$

II.2) (*) Many properties of an operator are not really reflected in its matrix representation.

- (i) Give an example of an operator T whose matrix in some basis only has non-zero elements in the diagonal, but T is not invertible.
- (ii) Give an example of an operator T whose matrix in some basis only has zero elements in the diagonal, but T is invertible.

II.3) (*) Determine if the following are true or false and think of a brief explanation of why that is the case.

- (i) Let $T : V \to V$ be a linear operator and assume that there is a basis B_V such that $\mathcal{M}(T, B_V)$ only has zeros on the diagonal. Then T is not invertible.
- (ii) Let V be a finite-dimensional inner product space and $T : V \to V$ an operator. Assume that there exists a basis B_V such that $\mathcal{M}(T, B_V) = \mathcal{M}(T, B_V)^\dagger$, i.e. the matrix representing T is equal to its conjugate transpose, then T is self-adjoint.
- (iii) Let V be a finite-dimensional inner product space and $T : V \to V$ an operator. Assume that there exists an *orthonormal basis* B_V such that $\mathcal{M}(T, B_V) = \mathcal{M}(T, B_V)^\dagger$, i.e. the matrix representing T is equal to its conjugate transpose, then T is self-adjoint.
- (iv) Let $T : V \to V$ and $S : V \to V$ be two operators, if $\mathcal{M}(T, B_V) = \mathcal{M}(S, B_V)$ for some basis B_V, then $T = S$.
- (v) There exists an invertible operator $T : V \to V$ on a finite-dimensional vector space such that there exists a basis B_V, such that $\mathcal{M}(T, B_V)$ is not invertible.

III. **Quotients and Duals**

III.1) (*) Let V be a finite-dimensional vector space and $U_1 \subseteq U_2$ two subspaces.

- (i) Prove that there is a surjective linear map $V/U_1 \to U/U_2$.
- (ii) Prove that there is an injective linear map $U_2/U_1 \to V/U_1$.
- (iii) Prove that $\dim(V/U_1) = \dim(U_2/U_1) + \dim(V/U_2)$.

III.2) (*) Let V be a vector space and $U \subseteq V$ a subspace. Assume that U is finite-dimensional, prove that V is isomorphic to $U \times V/U$.

III.3) (*) Let $U_1, U_2 \subseteq V$ be two subspaces, such that $U_1 \cap U_2 = \{0\}$. Prove that $(U_1 \oplus U_2)^\vee \simeq U_1^\vee \oplus U_2^\vee$. Explain what the direct sum means on each side of the equation.

III.4) (**) Let V be a finite-dimensional vector space and consider $\lambda_1, \lambda_2, \lambda_3 \in V^\vee$. Consider the following subspaces:

(1) $\operatorname{Span}\{\lambda_1, \lambda_2, \lambda_3\}$;

(2) $(\operatorname{null}\lambda_1 \cap \operatorname{null}\lambda_2 \cap \operatorname{null}\lambda_3)^0$;

(3) $\{\lambda \in V^\vee \mid \operatorname{null}\lambda_1 \cap \operatorname{null}\lambda_2 \cap \operatorname{null}\lambda_3 \subseteq \operatorname{null}\lambda\}$.

Prove that these three subspaces are equal. Please state clearly what implications you are proving at every step.

III.5) (*) Consider V a finite-dimensional vector space and let $V^\vee$ be its dual vector space. Let $B_{V^\vee} := \{\lambda_1, \ldots, \lambda_n\}$ be a basis of $V^\vee$. Prove that there exists a basis of V such that its dual basis is $B_{V^\vee}$.

III.6) Consider V and W two finite-dimensional vector spaces.

(i) Prove that $\mathcal{L}(V, W) \to \mathcal{L}(W^\vee, V^\vee)$ given by $T \mapsto T^\vee$ is an isomorphism of vector spaces.

(ii) Prove that T is invertible if and only if $T^\vee$ is invertible.

III.7) (*) Determine if the following are true or false and think of a brief explanation of why that is the case.

(i) For any vector space V and $V^\vee$ are isomorphic.

(ii) One always has $(V \times W)^\vee \simeq V^\vee \times W^\vee$.

(iii) For every $T : V \to W$, there exists an unique factorization

$$\begin{array}{ccc} V & \xrightarrow{\pi} & V/\operatorname{null} T \\ & {\scriptstyle T}\searrow & \downarrow{\scriptstyle S} \\ & & W \end{array},$$

i.e. an unique linear map S, such that the diagram above commutes.

(iv) Let $U \subseteq V$ be a subspace such that both U and V are infinite-dimensional. Then V/U is finite-dimensional.

IV. **Invariant subspaces and Minimal Polynomial**

IV.1) Let $T \in \mathcal{L}(V)$ on a finite-dimensional vector space and assume that there exists $v \in V$ such that $T^2v + 2Tv = -2v$.

(i) Assume that $\mathbb{F} = \mathbb{R}$, then prove that there does not exists a basis of V such that the matrix representing T in such a basis is upper-triangular.

(ii) Assume that $\mathbb{F} = \mathbb{C}$, then prove that if A is an upper-triangular matrix representing T in some basis, then $1 + i$ and $1 - i$ appear in the diagonal of A.

IV.2) (*) Let $T \in \mathcal{L}(V)$ and $\{v_1, \ldots, v_n\}$ be a basis of V. Prove that the following are equivalent:

(1) The matrix of T with respect to $\{v_1, \ldots, v_n\}$ is lower-triangular.

(2) $\operatorname{Span}\{v_k, \ldots, v_n\}$ is invariant under T for every $k \in \{1, \ldots, n\}$.

(3) $Tv_k \in \operatorname{Span}\{v_k, \ldots, v_n\}$ for every $k \in \{1, \ldots, n\}$.

Prove that over $\mathbb{C}$ every operator has a basis with respect to which it is lower-triangular.

IV.3) (*) Let $T : V \to V$ be an operator on a finite-dimensional vector space. Prove that the following are equivalent:

(1) $V = \operatorname{null} T \oplus \operatorname{range} T$;

(2) $\operatorname{null} T = \operatorname{null} T^2$;

(3) (?) $\operatorname{range} T = \operatorname{range} T^2$;

(4) $V = \operatorname{null} T + \operatorname{range} T$;

(5) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

IV.4) Let V and W be finite-dimensional vector spaces and consider $T_V \in \mathcal{L}(V)$ and $T_W \in \mathcal{L}(W)$. Assume that the only T_V-invariant subspaces of V are V and $\{0\}$ and similarly that the only T_W-invariant subspaces of W are W and $\{0\}$. Let $\alpha : V \to W$ be such that $\alpha \circ T_V = T_W \circ \alpha$. Prove that $\alpha = 0$ or α is an isomorphism.

IV.5) Let V be a finite-dimensional vector space. Prove that $\mathcal{L}(V)$ has a basis consisting of diagonalizable operators.

IV.6) Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$.

(i) (*) Prove that

$$\operatorname{Span}\{v, \dots, T^m v\} = \operatorname{Span}\{v, \dots, T^{\dim V - 1} v\}$$

for every $m \geq \dim V - 1$.

(ii) Prove that the minimal polynomial of T has degree at most $1 + \dim \operatorname{range} T$.

(iii) Prove that T is invertible if and only if $\mathrm{Id}_V \in \operatorname{Span}\{T, \dots, T^{\dim V}\}$.

IV.7) Determine if the following are true or false and think of a brief explanation of why that is the case.

(i) (*) Let $T, S : V \to V$ be two operators such that $TS = ST$. Let v be an eigenvector of T with eigenvalue λ. Then v is an eigenvector of S with eigenvalue λ.

(ii) Let $\{v_1, \dots, v_k\}$ be a basis of $\operatorname{range} T$ then $\{Tv_1, \dots, Tv_k\}$ contains a basis of $\operatorname{range} T^2$.

(iii) (*) Let $\{v_1, \dots, v_k\}$ be a sequence of eigenvectors for distinct eigenvalues $\{\lambda_1, \dots, \lambda_k\}$, consider $\alpha \in \mathbb{F}$ such that $\alpha \neq \lambda_i$ for every $i \in \{1, \dots, n\}$. If $\{v_1, \dots, v_k\}$ are linearly independent, then $\{(\alpha - \lambda_1)v_1, \dots, (\alpha - \lambda_k)v_k\}$ are linearly independent.

(iv) (*) Let $T : V \to V$ be an operator on a finite-dimensional vector space and assume that T is not diagonalizable. Then T^2 is also not diagonalizable.

(v) (*) Let $T : V \to V$ be an operator on a *complex* finite-dimensional vector space. Then T is diagonalizable if and only if there exists some positive $k \geq 1$ such that T^k is diagonalizable.

(vi) Let $T : V \to V$ be a diagonalizable and $U \subseteq V$ a subspace. Then $T/U : V/U \to V/U$ the operator induced on the quotient is diagonalizable.

(vii) (*) Let $T : V \to V$ be a diagonalizable and $U \subseteq V$ a T-invariant subspace. Then $T|_U : U \to U$ is diagonalizable.

(viii) Let $T : V \to V$ be an operator on a finite-dimensional vector space and $U \subseteq V$ a subspace such that T/U and $T|_U$ are diagonalizable. Then T is diagonalizable.

V. **Inner Product and Spectral Theorem**

V.1) Let $T : V \to V$ be an operator on a complex finite-dimensional vector space.

(i) (*) Suppose that T is normal and has real eigenvalues. Prove that T is self-adjoint.

(ii) (**) Show that any normal operator T is a product of S and R, where S is a self-adjoint operator and R is an operator all of whose (possibly complex) eigenvalues have absolute value 1.

V.2) Consider $V = \mathbb{C}^4$ with the standard inner product. Let $U = \operatorname{Span}\{(1,0,1,0),(0,1,1,0),(1,1,1,1)\}$.

(i) Find an orthonormal basis for U.

(ii) Calculate $P_U : V \to V$ the projection onto U.

(iii) (*) Is P_U normal or self-adjoint? If so, what does the spectral theorem applied to P_U give?

V.3) Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by $T(v) := \langle v, u \rangle x$ for every $v \in V$.

(i) Assume that V is a real inner product space. Prove that T is self-adjoint if and only if u and x are linearly dependent.

(ii) Prove that T is normal if and only if u and x are linearly dependent.

V.4) (*) Let V be an inner product space and $T \in \mathcal{L}(V)$.

(i) Assume that V is a real inner product space. Prove that T is self-adjoint if and only if (a) $V = \oplus_{i=1}^m E(\lambda_i, T)$ and (b) all pairs of eigenvectors corresponding to different eigenvalues are orthogonal.

(ii) Assume that V is a complex inner product space. Prove that T is normal if and only if (a) $V = \oplus_{i=1}^m E(\lambda_i, T)$ and (b) all pairs of eigenvectors corresponding to different eigenvalues are orthogonal.

(iii) How do the statements above change if one requires only one of the two conditions (a) or (b)?

V.5) Let $T : U \to V$ be a linear map between finite-dimensional inner product spaces.

(i) (*) Prove that

$$\dim \operatorname{null} T - \dim \operatorname{null} T^* = \dim U - \dim V.$$

(ii) Let $S : V \to W$ be another linear map. Define $R := TT^* + S^*S : V \to V$, assume that $\operatorname{range} T = \operatorname{null} S$. Show that R is invertible.

V.6) (*) Determine if the following are true or false and think of a brief explanation of why that is the case.

(i) Every orthogonal set is linearly independent.

(ii) Every orthonormal set is linearly independent.

(iii) Let $T : V \to V$ be an operator on a finite-dimensional inner product space and T^* its adjoint. Then v is an eigenvector of T if and only if v is an eigenvector of T^*.

(iv) Let $T : V \to V$ be an operator on a finite-dimensional real inner product space, such that $V = \operatorname{null} T \oplus \operatorname{range} T$, then T is self-adjoint.

(v) Let $T : V \to V$ be a normal operator on a finite-dimensional complex inner product space, then $V = \operatorname{null} T + \operatorname{range} T$.

VI. Generalized Eigenvalues and Eigenvectors, Jordan form

VI.1) (*) Let $T \in \mathcal{L}(V), \lambda \in \mathbb{F}$ and $m \geq 1$ an integer.

(i) Prove that $\dim \operatorname{null} T^m \leq m \dim \operatorname{null} T$.

(ii) Is $\dim \operatorname{null}(T - \lambda \operatorname{Id}_V)^m \geq m$? What if you assume that $(z - \lambda)^m$ is a factor of the minimal polynomial of T?

(iii) Can you formulate and prove similar claims to (i) and (ii) for $\operatorname{range} T^m$ and $\operatorname{range}(T - \lambda \operatorname{Id}_V)^m$?

VI.2) (*) Let $T : \mathbb{C}^6 \to \mathbb{C}^6$ be an operator with minimal polynomial $p_T(x) = (x-2)^2(x+1)^2$.

(i) Determine all possible Jordan forms of T.

(ii) Calculate the characteristic polynomial of each form in (i).

VI.3) Let $p, q \in \mathbb{C}[x]$ be two monic polynomials, with the same zeros and such that q is a multiple of p. Prove that there exists $T \in \mathcal{L}(\mathbb{C}^{\deg q})$ such that $c_T = q$ and $p_T = p$, i.e. the characteristic polynomial of T is q and the minimal polynomial of T is p.

VI.4) Let $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

(1) there does not exist two non-zero T-invariant subspaces $U, W \subseteq V$ such that $V = U \oplus W$.

(2) the minimal polynomial of T is $p_T(z) = (z - \lambda)^{\dim V}$, for some $\lambda \in \mathbb{C}$.

What happens if $\mathbb{F} = \mathbb{R}$?

VI.5) Consider the matrix:

$$A_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 1 & 0 \end{pmatrix}.$$

(i) Calculate the Jordan canonical form of A_ϵ when $\epsilon \neq 0$.

(ii) Calculate the Jordan canonical form of A_ϵ when $\epsilon = 0$.

VI.6) (*) Determine if the following are true or false and think of a brief explanation of why that is the case.

(i) Given $S, T \in \mathcal{L}(V)$ two nilpotent operators, then ST is nilpotent.

(ii) Let $T \in \mathcal{L}(V)$ be nilpotent and diagonalizable, then $T = 0$.

(iii) Given $S, T \in \mathcal{L}(V)$ two nilpotent operators, then $S + T$ is nilpotent.

(iv) Given $S, T \in \mathcal{L}(V)$ two nilpotent operators such that $ST = TS$, then $S + T$ and ST are nilpotent.

(v) Let $T \in \mathcal{L}(V)$, assume that there exists B_V a basis of V such that $\mathcal{M}(T, B_V)$ is a diagonal matrix. Then for any Jordan basis B'_V the matrix $\mathcal{M}(T, B'_V)$ is diagonal.

(vi) Let $T \in \mathcal{L}(V)$ on a complex vector space and consider B_V and B'_V two different Jordan basis. Then $\mathcal{M}(T, B_V)$ and $\mathcal{M}(T, B'_V)$ can have a different number of blocks in its diagonal form.

VII. **Tensor Product, Determinant, and Trace**

VII.1) (*) Let $\{v_1, \ldots, v_n\} \subset V$ and $\{w_1, \ldots, w_n\} \subset W$ be two lists of vectors.

(i) Assume that $\{v_1, \ldots, v_n\}$ are linearly independent and that $v_1 \otimes w_1 + \cdots + v_n \otimes w_n = 0$. Prove that $w_1 = \cdots = w_n = 0$.

(ii) Let $n = 3$, give an example to show that (i) fails if $\{v_1, \ldots, v_n\}$ is not linearly independent.

(iii) Assume that $\dim V > 1$ and $\dim W > 1$. Prove that $\{v \otimes w \mid v \in V,\ w \in W\} \neq V \otimes W$.

(iv) Explain why the condition on the dimensions of V and W in (iii)) is necessary.

VII.2) (*) Let V be a real vector space and $T \in \mathcal{L}(V)$.

(i) Assume that T has no eigenvalues, prove that $\det T > 0$.

(ii) Assume that $\dim V$ is even and that $\det T < 0$. Prove that T has at least two distinct eigenvalues.

VII.3) Let V be an inner product space and $T \in \mathcal{P}(V)$.

(i) Prove that $\operatorname{tr} T = \operatorname{tr} T^*$.

(ii) Assume that $T^2 = T$. Prove that $\operatorname{tr} T = \dim \operatorname{range} T$.

VII.4) Let $T \in \mathcal{L}(V)$ on a finite-dimensional vector space.

(i) Consider $T^\vee \in \mathcal{L}(V^\vee)$ the dual operator determined by T. Prove that $\det T = \det T^\vee$.

(ii) Assume that V is an inner product space. Prove that $\det T^* = \overline{\det T}$, where $T^* \in \mathcal{L}(V)$ is the adjoint operator.

VII.5) (*) Consider $f : V_1 \to V_2$ and $g : U_1 \to U_2$ linear maps between finite-dimensional vector spaces.

(i) Assume that f and g are surjective, prove that $f \otimes g$ is surjective.

(ii) Assume that f and g are injective, prove that $f \otimes g$ is injective.

VII.6) (*) Determine if the following are true or false and think of a brief explanation of why that is the case.

(i) Let $T, S \in \mathcal{L}(V)$ be two operators on a finite-dimensional vector space. Then $\operatorname{tr}(TS) = \operatorname{tr}(T)\operatorname{tr}(S)$.

(ii) Let V be a finite-dimensional vector space. There exist $T, S \in \mathcal{L}(V)$ such that $ST - TS = \mathrm{Id}_V$.

(iii) Let $T \in \mathcal{L}(V)$ be an operator on a finite-dimensional vector space. Assume that $\operatorname{tr}(TS) = 0$ for all $S \in \mathcal{L}(V)$, then $T = 0$.

(iv) Consider U, V, W then $(U \oplus V) \otimes W \simeq U \otimes W \oplus V \otimes W$.

(v) Assume that T is nilpotent, then $\det(\mathrm{Id} + T) = 1$.

(vi) Let $T, S \in \mathcal{L}(V)$ be two operators on a finite-dimensional vector space. Then $\det(T + S) = \det(T) + \det(S)$.