Math 2102: Homework 5
Due on: Apr. 29 at 11:59 pm.

All assignments must be submitted via Moodle. Precise and adequate explanations should be given to each problem. Exercises marked with Extra might be more challenging or a digression, so they won't be graded.

1. Let $n \geq 1$. Consider $\beta : V^{\times n} \to \mathbb{F}$ an n-linear map.
   (i) Let $\alpha : V^{\times n} \to \mathbb{F}$ be defined by
   $$\alpha(v_1, \ldots, v_n) := \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$
   Prove that $\alpha \in \mathcal{L}^n_{\text{alt}}(V)$, i.e. α is an alternating n-linear map.
   (ii) Let $\alpha : V^{\times n} \to \mathbb{F}$ be defined by
   $$\alpha(v_1, \ldots, v_n) := \sum_{\sigma \in S_n} \beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$
   Prove that $\alpha \in \mathcal{L}^n_{\text{sym}}(V)$, i.e. α is a symmetric n-linear map.
   (iii) Give an example of an alternating 2-linear map α on $\mathbb{R}^3$ such that there are linearly independent vectors v_1, v_2 in $\mathbb{R}^3$ such that $\alpha(v_1, v_2) \neq 0$.

2. Let $T \in \mathcal{L}(V)$ be an operator.
   (i) Assume that T has no eigenvalues. Prove that $\det T > 0$.
   (ii) Suppose that V is a real vector space of odd-dimension. Without using the minimal polynomial, prove that T has an eigenvalue.

3. Given vector spaces V, V', V'' we say that a composition of morphisms:
   $$V' \xrightarrow{\imath_V} V \xrightarrow{p_V} V''$$
   is an *exact sequence*, if it satisfies:
   a) $\operatorname{null} \imath_V = \{0\}$;
   b) $\operatorname{range} \imath_V = \operatorname{null} p_V$;
   c) $\operatorname{range} p_V = V''$.

   Consider two exact sequences $V' \xrightarrow{\imath_V} V \xrightarrow{p_V} V''$ and $U' \xrightarrow{\imath_U} U \xrightarrow{p_V} U''$.
   (i) Prove that one has an exact sequence:
   $$V' \oplus U' \xrightarrow{\imath_V \oplus \imath_U} V \oplus U \xrightarrow{p_U \oplus p_V} V'' \oplus U''.$$
   (ii) Prove that there are linear maps:
   $$V' \otimes U' \xrightarrow{\imath_V \otimes \imath_U} V \otimes U \xrightarrow{p_V \otimes p_U} V'' \otimes U''. \tag{1}$$
   (iii) Is the sequence (1) exact? What fails? Consider the cases of V and U trivial, one-dimensional and with dimension(s) greater than two to understand the general answer.