Math 2102: Homework 5
Solutions

1. Let $n \geq 1$. Consider $\beta : V^{\times n} \to \mathbb{F}$ an n-linear map.

   (i) Let $\alpha : V^{\times n} \to \mathbb{F}$ be defined by

   $$\alpha(v_1, \ldots, v_n) := \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

   Prove that $\alpha \in \mathcal{L}^n_{\text{alt}}(V)$, i.e. α is an alternating n-linear map.

   Solution. *Let $\tau \in S_n$ we have:*

   $$\begin{aligned}\tau^*\alpha(v_1, \ldots, v_n) &= \alpha(v_{\tau(1)}, \ldots, v_{\tau(n)})\\ &= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\beta(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(n)})\\ &= (-1)^\tau \sum_{\sigma \in S_n} \operatorname{sign}(\sigma\tau)\beta(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(n)})\\ &= (-1)^\tau \sum_{\sigma' \in S_n} \operatorname{sign}(\sigma')\beta(v_{\sigma'(1)}, \ldots, v_{\sigma'(n)})\\ &= (-1)^\tau \alpha(v_1, \ldots, v_n),\end{aligned}$$

   where in the fourth equality we used the change of variables $\sigma \mapsto \sigma' := \sigma\tau$, which gives the same sum, since $(-)\tau : S_n \to S_n$ is a bijection.

   (ii) Let $\alpha : V^{\times n} \to \mathbb{F}$ be defined by

   $$\alpha(v_1, \ldots, v_n) := \sum_{\sigma \in S_n} \beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}).$$

   Prove that $\alpha \in \mathcal{L}^n_{\text{sym}}(V)$, i.e. α is a symmetric n-linear map.

   Solution. *Let $\tau \in S_n$ we have:*

   $$\begin{aligned}\tau^*\alpha(v_1, \ldots, v_n) &= \alpha(v_{\tau(1)}, \ldots, v_{\tau(n)})\\ &= \sum_{\sigma \in S_n} \beta(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(n)})\\ &= \sum_{\sigma' \in S_n} \beta(v_{\sigma'(1)}, \ldots, v_{\sigma'(n)})\\ &= \alpha(v_1, \ldots, v_n),\end{aligned}$$

   where in the third equality is justified as in (i).

   (iii) Give an example of an alternating 2-linear map α on $\mathbb{R}^3$ such that there are linearly independent vectors v_1, v_2 in $\mathbb{R}^3$ such that $\alpha(v_1, v_2) \neq 0$.

   Solution. *We define $\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $\alpha((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1y_2 - x_2y_1$. Notice that this is alternating, since $\alpha((y_1, y_2, y_3), (x_1, x_2, x_3)) = -(x_1y_2 - x_2y_1)$. We have*

   $$\alpha((1, 0, 0), (0, 1, 0)) = 1.$$

2. Let $T \in \mathcal{L}(V)$ be an operator.

   (i) Assume that T has no eigenvalues. Prove that $\det T > 0$.

   Solution. *Assume by contradiction that* $\det T < 0$. *Consider* T *as an operator with complex coefficients, i.e.* $T_{\mathbb{C}}$. *We know that* $T_{\mathbb{C}}$ *has eigenvalues, as an operator over a finite-dimensional complex vector space and that* $\det T = \det(T_{\mathbb{C}}) = \prod_{i=1}^{n} \lambda_i$. *Since* $\det T$ *is real, up to reordering we have that the product of eigenvalues* λ_i *can be split into two parts:*

$$\det T = \prod_{i=1}^{m} \lambda_i \prod_{j=1}^{(n-m)/2} \lambda_{2j-1+m}\lambda_{2j+m},$$

   where $\{\lambda_1, \ldots, \lambda_m\} \subset \mathbb{R}$ *and* $\{\lambda_{m+1}, \ldots, \lambda_n\}$ *are complex with* $\lambda_{\lambda_{2j-1+m}} = \overline{\lambda_{2j+m}}$. *This implies that*

$$\det T = c \prod_{i=1}^{m} \lambda_i,$$

   for some $c \geq 0$. *Since* $\det T < 0$, *we have that* $c > 0$. *Now if* $T_{\mathbb{C}}$ *had no real eigenvalues, then* $\prod_{i=1}^{m} \lambda_i = 1$, *which would imply that* $\det T > 0$. *So there exists* $\lambda \in \mathbb{R}$ *such that* λ *is an eigenvalue of* $T_{\mathbb{C}}$, *so* λ *is an eigenvalue of* T*,by HW 2, Exercise 3 (i).*

   (ii) Suppose that V is a real vector space of odd-dimension. Without using the minimal polynomial, prove that T has an eigenvalue.

   Solution. *There are two cases, consider* $\det T = 0$. *In this case,* T *is not injective, which implies that* 0 *is an eigenvalue.*

   So we can consider $\det T = a \neq 0$. *If* $a < 0$, *then (i) gives that* T *has an eigenvalue. If* $a > 0$, *we notice that (by Example 43 (ii) in the Lecture Notes)*

$$\det(-T) = (-1)^{\dim V} \det T < 0.$$

   So $-T$ *has an eigenvalue, i.e.* $-Tv = \lambda v$ *for some non-zero* $v \in V$, *this implies that* $-\lambda$ *is an eigenvalue of* T. *And we are done.*

3. Given vector spaces V, V', V'' we say that a composition of morphisms:

$$V' \xrightarrow{\imath_V} V \xrightarrow{p_V} V'' \tag{1}$$

   is an *exact sequence*, if it satisfies:

   a) $\operatorname{null} \imath_V = \{0\}$;

   b) $\operatorname{range} \imath_V = \operatorname{null} p_V$;

   c) $\operatorname{range} p_V = V''$.

   Consider two exact sequences $V' \xrightarrow{\imath_V} V \xrightarrow{p_V} V''$ and $U' \xrightarrow{\imath_U} U \xrightarrow{p_V} U''$.

   (i) Prove that one has an exact sequence:

$$V' \oplus U' \xrightarrow{\imath_V \oplus \imath_U} V \oplus U \xrightarrow{p_U \oplus p_V} V'' \oplus U''.$$

Solution. *For a), let* $x \in V' \oplus U'$ *then* $x = v + u$*, with* $v \in V'$ *and* $u \in U'$ *such that*

$$\imath_V \oplus \imath_U(x) = \imath_V(v) + \imath_U(u) = 0.$$

Since $\imath_V(v) \in V$ *and* $\imath_U(u) \in U$ *we obtain that* $\imath_V(v) = 0$ *and* $\imath_U(u) = 0$*, which since* $\imath_V$ *and* $\imath_U$ *ar injective gives that* $v = 0$ *and* $u = 0$*, so* $x = 0$*.*
For c), let $x \in V'' \oplus U''$ *given by* $x = v + u$*, with* $v \in V''$ *and* $u \in U''$*. Let* $\tilde{v} \in V$ *and* $\tilde{u}$ *such that* $p_V(\tilde{v}) = v$ *and* $p_U(\tilde{u}) = u$*, then we compute:*

$$p_V \oplus p_U(\tilde{v} + \tilde{u}) = p_V(\tilde{v}) + p_U(\tilde{u}) = v + u = x.$$

So $p_V \oplus p_U$ *is surjective.*
For b), we need to prove both inclusions. Let $x \in \operatorname{range}(\imath_V \oplus \imath_U)$*, then* $x = \imath_V(v) + \imath_U(u)$ *for some* $v \in V'$ *and* $u \in U'$*. We compute:*

$$p_V \oplus p_U(\imath_V(v) + \imath_U(u)) = p_V \circ \imath_V(v) + p_U \circ \imath_U(u) = 0,$$

where the middle equality is justified because $\imath_V(v) \in V$ *and* $\imath_U(u) \in U$*. Let* $x \in \operatorname{null}(p_V \oplus p_U)$*, then* $x = v + u$ *for some* $v \in V$ *and* $u \in U$ *such that*

$$p_V \oplus p_U(v + u) = p_V(v) + p_U(u) = 0.$$

Since $p_V(v) \in V''$ *and* $p_U(u) \in U''$ *we have that* $p_V(v) = 0 = p_U(u)$*. Thus, by assumption there exists* $\tilde{v} \in V'$ *and* $\tilde{u} \in U'$ *such that* $v = \imath_V(\tilde{v})$ *and* $u = \imath_U(\tilde{u})$*. Then we see that*

$$(\imath_V \oplus \imath_U)(\tilde{v} + \tilde{u}) = \imath_V(\tilde{v}) + \imath_U(\tilde{u}) = v + u = x.$$

This finishes the proof.

(ii) Prove that there are linear maps:

$$V' \otimes U' \xrightarrow{\imath_V \otimes \imath_U} V \otimes U \xrightarrow{p_V \otimes p_U} V'' \otimes U''. \tag{2}$$

Solution. *This follows from Exercise 4 (i) in Tutorial 8.*

(iii) Is the sequence (2) exact? What fails? Consider the cases of V and U trivial, one-dimensional and with dimension(s) greater than two to understand the general answer.

Solution. *First we prove the following. Given an exact sequence as (1), then*

$$\dim V = \dim V'' + \dim V'. \tag{3}$$

Indeed, by the fundamental theorem applied to p_V *and* $\imath_V$ *we obtain:*

$$\dim V = \dim \operatorname{null} p_V + \dim \operatorname{range} p_V \quad \textit{and} \quad \dim V' = \dim \operatorname{null} \imath_V + \dim \operatorname{range} \imath_V.$$

Now a) gives that $\dim \operatorname{null} \imath_V = 0$*, b) that* $\dim \operatorname{null} p_V = \dim \operatorname{range} \imath_V$ *and c) that* $\dim \operatorname{range} p_V = V''$*. Combining all of this gives (3).*
Thus, if (2) were exact we would obtain:

$$\dim V \dim U = \dim(V \otimes U) = \dim V' \dim U' + \dim V'' \dim U''.$$

Since $\dim V = \dim V' + \dim V''$ *and* $\dim U = \dim U' + \dim U''$*. We obtain:*

$$\dim V' \dim U' + \dim V'' \dim U'' + \dim V' \dim U'' + \dim V'' \dim U' = \dim V' \dim U' + \dim V'' \dim U''. \tag{4}$$

Notice that if either $\dim V' \dim U''$ *or* $\dim V'' \dim U'$ *are non-zero, then we obtain a contradiction.*

In fact, we can prove that a) and c) always hold by considering bases of these vector spaces. So the main problem is b).

If $\dim V = 1$*, then either (1)* $V' \simeq V$*, which gives that (2) is exact if and only if* $U' \simeq U$ *or (2)* $V \simeq V''$*, which gives that (2) is exact if and only if* $U \simeq U''$*. If either of* V *or* U *are trivial, then (2) is also exact, since it is simply the sequence* $\{0\} \xrightarrow{0} \{0\} \xrightarrow{0} \{0\}$*.*