Math 2102: Homework 4
Solutions

1. Let $T, S : V \to V$ be two operators on a complex finite-dimensional inner product space. Assume that $TS = ST$.

   (i) Prove that there is an orthonormal basis of V with respect to which T and S are upper-triangular.

   Solution. *By Exercise 2. (iii) from HW 3 for the set* $\mathcal{E} := \{S, T\} \subset \mathcal{L}(V)$, *we know that there exists a basis* $B_V = \{v_1, \ldots, v_n\}$ *such that* $\mathcal{M}(T, B_V)$ *and* $\mathcal{M}(S, B_V)$ *are both upper-triangular, i.e.*

   $$T(v_i) \in \operatorname{Span}\{v_1, \ldots, v_i\} \quad \text{and} \quad S(v_i) \in \operatorname{Span}\{v_1, \ldots, v_i\}$$

   for every $i \in \{1, \ldots, n\}$.
   Let $B'_V = \{e_1, \ldots, e_n\}$ *be the basis obtained from* B_V *via the Gram–Schmidt procedure. We prove by induction on* i *that* $T(e_i) \in \operatorname{Span}\{e_1, \ldots, e_i\}$ *and* $S(e_i) \in \operatorname{Span}\{e_1, \ldots, e_i\}$.
   For $i = 1$, *since* $e_1 := \frac{1}{\|v_1\|} v_1$, *we have that*

   $$T(e_1) = \frac{1}{\|v_1\|} T(v_1) \in \operatorname{Span}\{v_1\} = \operatorname{Span}\{e_1\}$$

   and similarly for S.
   Now assume that we proved the claim for every $i < k$. *Notice that*

   $$T(e_k) = T(v_k - u_k) = T(v_k) - T(u_k),$$

   where $u_k \in \operatorname{Span}\{e_1, \ldots, e_{k-1}\}$ *and* $T(u_k) \in \operatorname{Span}\{e_1, \ldots, e_{k-1}\}$ *by the inductive hypothesis. By Theorem 3 in the Lecture Notes, we have*

   $$T(v_k) \in \operatorname{Span}\{v_1, \ldots, v_k\} = \operatorname{Span}\{e_1, \ldots, e_k\},$$

   which gives that $T(e_k) \in \operatorname{Span}\{e_1, \ldots, e_k\}$. *Exactly the same argument applies to* S. *So we are done.*

   (ii) Assume that T is normal. Use (i) to give a different proof of the complex spectral theorem.

   Solution. *Let* T *be a normal operator. By Exercise 2. (iii) from HW 3 and (i) above there exists an orthonormal basis* B_V *such that* $\mathcal{M}(T, B_V)$ *and* $\mathcal{M}(T^*, B_V)$ *are upper-triangular. Since* B_V *is orthonormal, we have:*

   $$\mathcal{M}(T^*, B_V) = \mathcal{M}(T, B_V)^\dagger$$

   which implies that $\mathcal{M}(T, B_V)$, *and hence* $\mathcal{M}(T^*, B_V)$ *are diagonal.*

2. Let $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{C}$. Recall the definition of $T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ from Exercise 3 in HW 2.

   (i) Show that $u + iv \in G(\lambda, T_{\mathbb{C}})$ if and only if $u - iv \in G(\overline{\lambda}, T_{\mathbb{C}})$.

   Solution. *Let* $u + iv \in V$, *we prove by induction on* k, *that*

   $$(T_{\mathbb{C}} - \lambda)^k (u + iv) = 0 \;\Rightarrow\; (T_{\mathbb{C}} - \overline{\lambda})^k (u - iv) = 0.$$

   For $k = 1$, *this is Exercise 3 (ii) from HW2.*

Let $(T_{\mathbb{C}} - \lambda)(u + iv) = u' + iv'$*, then we have that*

$$(T_{\mathbb{C}} - \lambda)^{k-1}(u' + iv') = 0,$$

which implies that

$$(T_{\mathbb{C}} - \overline{\lambda})^{k-1}(u' - iv') = 0.$$

Since $\lambda \in \mathbb{R}$ *we have* $u' = T(u) - \lambda u$ *and* $v' = T(v) - \lambda v$*, so we obtain:*

$$(T_{\mathbb{C}} - \overline{\lambda})^{k-1}(u' - iv') = (T_{\mathbb{C}} - \overline{\lambda})^{k-1}(T(u) - \lambda u - i(T(v) - \lambda v)) = (T_{\mathbb{C}} - \overline{\lambda})^{k}(u - iv).$$

The other direction is proved in exactly the same way, which gives the conclusion of the question.

(ii) Show that the (algebraic) multiplicity of λ as an eigenvalue of $T_{\mathbb{C}}$ is the same as the (algebraic) multiplicity of $\overline{\lambda}$ as an eigenvalue of $T_{\mathbb{C}}$.

Solution. *Recall that the algebraic multiplicity* λ *in* $T_{\mathbb{C}}$ *is* $\dim G(\lambda, T_{\mathbb{C}})$*. Let* $\{v_1, \ldots, v_n\}$ *be a basis of* $G(\lambda, T_{\mathbb{C}})$*. For each* i*, we write:*

$$v_i = e_i + if_i, \quad \text{for } e_i, f_i \in V.$$

Let $\overline{v_i} := e_i - if_i$ *be vectors in* $\dim G(\overline{\lambda}, T_{\mathbb{C}})$*, notice that they belong to the generalized eigenspace of* $\overline{\lambda}$ *by (i). We claim that* $\{\overline{v_1}, \ldots, \overline{v_n}\}$ *is a basis of* $G(\overline{\lambda}, T_{\mathbb{C}})$*.*
Firstly, we show that $\{\overline{v_1}, \ldots, \overline{v_n}\}$ *span* $G(\overline{\lambda}, T_{\mathbb{C}})$*, let* $v + iu \in G(\overline{\lambda}, T_{\mathbb{C}})$*, then* $v - iu \in G(\lambda, T_{\mathbb{C}})$ *so there are* $a_1, \ldots, a_n \in \mathbb{C}$ *such that*

$$\begin{aligned} v - iu &= \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} (x_i + iy_i)(e_i + if_i) \\ &= \sum_{i=1}^{n} ((x_i e_i - y_i f_i) + i(y_i e_i + x_i f_i)), \end{aligned}$$

where $a_i = x_i + iy_i$*. So*

$$\begin{aligned} v + iu &= \sum_{i=1}^{n} ((x_i e_i - y_i f_i) - i(y_i e_i + x_i f_i)) \\ &= \sum_{i=1}^{n} (x_i - iy_i)(e_i - if_i) \\ &= \sum_{i=1}^{n} \overline{a_i v_i}. \end{aligned}$$

Secondly, we check that $\{\overline{v_1}, \ldots, \overline{v_n}\}$ *are linearly independent. Indeed, assume that there are* $a_1, \cdots, a_n \in \mathbb{C}$ *such that*

$$\sum_{i=1}^{n} a_i \overline{v_i} = \sum_{i=1}^{n} (x_i + iy_i)(e_i - if_i) = \sum_{i=1}^{n} ((x_i e_i + y_i f_i) + i(y_i e_i - x_i f_i)) = 0.$$

This implies that

$$\sum_{i=1}^{n} ((x_i e_i + y_i f_i) = 0 \quad \text{and} \quad \sum_{i=1}^{n} (y_i e_i - x_i f_i) = 0. \tag{1}$$

Consider the linear combination:

$$\sum_{i=1}^{n} \overline{a_i} v_i = \sum_{i=1}^{n} (x_i - iy_i)(e_i + if_i) = \sum_{i=1}^{n} ((x_i e_i + y_i f_i) - i(y_i e_i - x_i f_i)) = 0,$$

where the last equation comes by substituting (1) for the real and imaginary part. Since $\{v_1, \ldots, v_n\}$ *are linearly independent, we have that* $\overline{\alpha_1} = \cdots = \overline{\alpha_n} = 0$*, which gives that* $\alpha_1 = \cdots = \alpha_n = 0$*. This finishes the proof.*

(iii) Use (ii) to show that if $\dim V$ is an odd number, then $T_{\mathbb{C}}$ has a real eigenvalue.

Solution. *Assume that all eigenvalues of* λ *of* $T_{\mathbb{C}}$ *are complex, then by (ii) and the generalized eigenspace decomposition we have that:*

$$V_{\mathbb{C}} = \oplus_{i=1}^{m} G(\lambda_i, T_{\mathbb{C}}) \oplus \oplus_{i=1}^{m} G(\overline{\lambda_i}, T_{\mathbb{C}}).$$

Thus, we obtain that $\dim V_{\mathbb{C}} = \sum_{i=1}^{m} 2 \dim G(\lambda_i, T_{\mathbb{C}})$*. Since* $\dim V_{\mathbb{C}} = \dim V$*, notice that* $\dim V_{\mathbb{C}}$ *is the dimension of* $V_{\mathbb{C}}$ *as a* complex *vector space and* $\dim V$ *is the dimension of* V *as a* real *vector space. This is a contradiction with* $\dim V$ *being odd.*

(iv) Use (iii) to give an alternative proof of Proposition 6 in the Lecture Notes, namely that $\dim V$ is odd then T has an eigenvalue.

Solution. *Let* $T : V \to V$ *be an operator on an odd-dimensional real vector space, consider* $T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$*. By (iii),* $T_{\mathbb{C}}$ *has a real eigenvalue. By HW 2 Exercise 3 (i), we have that* λ *is an eigenvalue of* T*.*

3. Assume $\mathbb{F} = \mathbb{C}$ and consider $T \in \mathcal{L}(V)$ an operator on a finite-dimensional vector space. Prove that there does not exist a decomposition of V into a direct sum of two T-invariant subspaces if and only if the minimal polynomial of T is $(z - \lambda)^{\dim V}$ for some $\lambda \in \mathbb{C}$.

Solution. *First assume that there does not exist a decomposition of* V *into a direct sum of two* T*-invariant subspaces. Since we are over* $\mathbb{C}$ *we have that*

$$V = G(\lambda, T)$$

for a single eigenvalue $\lambda \in \mathbb{C}$*. Let*

$$\mathcal{M}(T, B_V) = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{pmatrix}$$

denote the Jordan form of T *for some Jordan basis* B_V*. If one of the blocks* A_i *is smaller than* n *by* n*, say* k *by* k *,we can consider* $\{v_{j_1}, \ldots, v_{j_k}\}$ *the vectors of the basis corresponding to this block and it is clear that* $V = U \oplus W$*, with* $U = \operatorname{Span}\{v_{j_1}, \ldots, v_{j_k}\}$ *and* W *spanned by the remaining elements of the basis. This implies that the matrix* A *representing* $T - \lambda$ *has only* 1*'s above the diagonal and* 0 *everywhere else. Let* $n = \dim V$*, clearly we have* $A^n = 0$ *and* $A^{n-1} \neq 0$*, which gives* $(T - \lambda)^n = 0$ *and* $(T - \lambda)^{n-1} \neq 0$*; that is the minimal polynomial has the form claimed.*

Suppose that $p_T(z) = (z - \lambda)^{\dim V}$*. Then the Jordan form of* T *has* λ *in the diagonal and* 1 *just above it, since otherwise* $(z - \lambda)^k = 0$ *for some* $k < \dim V$*. Assume that* $U \oplus W$ *is a decomposition of* V *into subspaces both of which are non-zero. Then* U *and* W *are also invariant under* $N := (T - \lambda)$*, which implies that* $N| : U \to U$ *and* $N|_W : W \to W$ *are both nilpotent, which gives that*

$$N^M = (N|_U \oplus N|_W)^M = (N|_U)^M \oplus (N|_W)^M = 0,$$

where $M = \max\{dimU, \dim W\} < \dim V$*; which is a contradiction with* p_T *being the minimal polynomial of* T*.*

4. Let V and W be two finite-dimensional inner product spaces.

(i) Prove that $\langle S, T\rangle := \operatorname{tr}(T^*S)$ determines an inner product on $\mathcal{L}(V,W) \times \mathcal{L}(V,W)$.

Solution. *By definition we have that* $\operatorname{tr}(\lambda T_1 + T_2) = \lambda \operatorname{tr}(T_1) + \operatorname{tr}(T_2)$. *Since* $(\lambda T)^* = \overline{\lambda} T^*$ *it is clear that*

$$\langle S_1 + \lambda S_2, T\rangle = \langle S_1, T\rangle + \lambda \langle S_2, T\rangle \quad \text{and} \quad \langle S, T_1 + \lambda T_2\rangle = \langle S, T_1\rangle + \overline{\lambda} \langle S, T_2\rangle .$$

We need to check that for any T, *we have* $\operatorname{tr}(TT^*) \geq 0$. *Indeed,*

$$\operatorname{tr}(TT^*) = \sum_{i=1}^{n} \langle e_i, TT^*(e_i)\rangle = \sum_{i=1}^{n} \|T^* e_i\|^2 \geq 0,$$

where in the first equality we used $\{e_1, \dots, e_n\}$ *some orthonormal basis of* V *and the formula for the trace in terms of matrix representation.*
Assume that $\operatorname{tr}(TT^*)$, *then we have* $\sum_{i=1}^{n} \|T^* e_i\|^2 = 0$, *which only happens if* $T^* e_i = 0$ *for each* e_i, *which implies that* T^*. *Thus,* $T = 0$ *by picking an orthonormal basis of* V.
Finally, by considering B_V *an orthonormal basis of* V *we have:*

$$\operatorname{tr}(T^*S) = \sum_{i=1}^{n} \mathcal{M}(T^*S, B_V)_{i,i} = \sum_{i=1}^{n} \overline{\mathcal{M}((T^*S)^*, B_V)_{i,i}} = \overline{\sum_{i=1}^{n} \mathcal{M}(S^*T, B_V)} = \operatorname{tr}(S^*T).$$

(ii) Let $B_V = \{e_1, \dots, e_n\}$ be an orthonormal basis of V and $B_W = \{f_1, \dots, f_m\}$ be an orthonormal basis of W. Let $\langle -, -\rangle_{\text{std}} : \mathbb{F}^{mn} \times \mathbb{F}^{mn} \to \mathbb{F}$ be the standard inner product on $\mathbb{F}^{mn}$ (i.e. Example 23 (i) and (ii) from the Lecture Notes). Let $\mathcal{M}(-, B_V, B_W) : \mathcal{L}(V,W) \xrightarrow{\sim} \mathbb{F}^{mn}$ be the isomorphism given by the matrix coefficients. Prove that

$$\langle S, T\rangle = \langle \mathcal{M}(S, B_V, B_W), \mathcal{M}(T, B_V, B_W)\rangle_{\text{std}} \tag{2}$$

for all $S, T \in \mathcal{L}(V,W)$.

Solution. *Let* $S, T \in \mathcal{L}(V,W)$ *and* $A := \mathcal{M}(S, B_V, B_W)$ *and* $B := \mathcal{M}(T, B_V, B_W)$ *be the matrices in* $\mathbb{F}^{m,n}$. *We compute the left-hand side of (2):*

$$\operatorname{tr}(T^*S) = \sum_{i=1}^{n} \sum_{j=1}^{m} A^{\dagger}_{i,j} B_{j,i}.$$

For the left-hand side of (2) we have:

$$\langle \mathcal{M}(S, B_V, B_W), \mathcal{M}(T, B_V, B_W)\rangle_{\text{std}} = \sum_{i=1}^{n} \sum_{j=1}^{m} B_{j,i} \overline{A_{j,i}}.$$

Since $A^{\dagger}_{i,j} = \overline{A_{j,i}}$, *we obtain that the equality we needed to prove.*