

Math 2102: Homework 3
Due on: Mar. 18, 2024 at 11:59 pm.

All assignments must be submitted via Moodle. Precise and adequate explanations should be given to each problem. Exercises marked with Extra might be more challenging or a digression, so they won't be graded.

1. Let $T : V \rightarrow V$ be an operator on a finite-dimensional complex vector space. Prove that the following are equivalent:
 - (1) T is diagonalizable (i.e. it satisfies Proposition 6 from the Lecture Notes);
 - (2) $V = \text{null}(T - \lambda \text{Id}_V) \oplus \text{range}(T - \lambda \text{Id}_V)$ for every $\lambda \in \mathbb{C}$;
 - (3) the minimal polynomial $p_T = \prod_{i=1}^m (z - \lambda_i)$, where $\{\lambda_1, \dots, \lambda_m\}$ are distinct;
 - (4) there does not exist $\lambda \in \mathbb{C}$ such that p_T is a multiple of $(z - \lambda)^2$;
 - (5) p_T and p'_T have no zeros in common;
 - (6) the **greatest common divisor** of p_T and p'_T is the constant polynomial 1.
2. Let V be a finite-dimensional vector space over \mathbb{C} . Let $\mathcal{E} \subseteq \mathcal{L}(V)$ be a subset of linear operators which commute, i.e. for any $S, T \in \mathcal{E}$ we have $ST = TS$.
 - (i) Prove that for any $S, T \in \mathcal{E}$ the subspaces $\text{null } p(S)$ and $\text{range } p(S)$ are invariant under T .
 - (ii) Prove that there is a vector in V that is an eigenvector for every element of \mathcal{E} .
 - (iii) Prove that there is a basis of V with respect to which every element of \mathcal{E} has an upper-triangular form.
3. Let V be a finite-dimensional vector space.
 - (i) Let $T \in \mathcal{L}(V)$ be an invertible operator and $B_V = \{v_1, \dots, v_n\}$ is a basis such that $\mathcal{M}(B_V, T)$ is upper triangular with $\lambda_1, \dots, \lambda_n$ on the diagonal. Show that $\mathcal{M}(B_V, T^{-1})$ is also upper triangular with $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ on the diagonal.
 - (ii) Give an example of $T \in \mathcal{L}(V)$ and B_V such that $\mathcal{M}(T, B_V)$ contains only 0's in the diagonal but T is invertible.
 - (iii) Give an example of $T \in \mathcal{L}(V)$ and B_V such that $\mathcal{M}(T, B_V)$ contains only non-zero elements in the diagonal but T is not invertible.