Math 2102: Homework 3
Solutions

1. Let $T : V \to V$ be an operator on a finite-dimensional complex vector space. Prove that the following are equivalent:

   (1) T is diagonalizable (i.e. it satisfies Proposition 7 from the Lecture Notes);

   (2) $V = \operatorname{null}(T - \lambda \operatorname{Id}_V) \oplus \operatorname{range}(T - \lambda \operatorname{Id}_V)$ for every $\lambda \in \mathbb{C}$;

   (3) the minimal polynomial $p_T = \prod_{i=1}^{m}(z - \lambda_i)$, where $\{\lambda_1, \ldots, \lambda_m\}$ are distinct;

   (4) there does not exist $\lambda \in \mathbb{C}$ such that p_T is a multiple of $(z - \lambda)^2$;

   (5) p_T and p_T' have no zeros in common;

   (6) the greatest common divisor of p_T and p_T' is the constant polynomial 1.

   Solution. *(1) ⇒ (2): let $\{\lambda_1, \ldots, \lambda_m\}$ be all the eigenvalues of T. We claim that* $\operatorname{null}(T - \lambda \operatorname{Id}_V) \cap \operatorname{range}(T - \lambda \operatorname{Id}_V) = \{0\}$. *Let $v \in \operatorname{null}(T - \lambda \operatorname{Id}_V) \cap \operatorname{range}(T - \lambda \operatorname{Id}_V)$, then $v \in E(\lambda_i, T)$ for some $i \in \{1, \ldots, m\}$. We have:*

   $$(T - \lambda)(v) = (\lambda_i - \lambda)v = 0,$$

   since $v \neq 0$, we get $\lambda = \lambda_i$. But if $v = (T - \lambda)(u)$ by the same reasoning we obtain $v = 0$, which is a contradiction.

   (2) ⇒ (1): by Proposition 4, there exist λ_1 an eigenvalue of T, i.e. $\operatorname{null}(T - \lambda_1 \operatorname{Id}_V) \neq \{0\}$. *Let* $T|_{\operatorname{range}(T-\lambda_1 \operatorname{Id}_V)} : \operatorname{range}(T - \lambda_1 \operatorname{Id}_V) \to \operatorname{range}(T - \lambda_1 \operatorname{Id}_V)$, *by Proposition 4, there exists λ_2 such that $E(\lambda_2, T|_{\operatorname{range}(T-\lambda_1 \operatorname{Id}_V)}) \neq \{0\}$. Proceeding like this we obtain a chain:*

   $$\begin{aligned} V &= \operatorname{null}(T - \lambda_1 \operatorname{Id}_V) \oplus \operatorname{range}(T - \lambda_1 \operatorname{Id}_V) \\ &= \operatorname{null}(T - \lambda_1 \operatorname{Id}_V) \oplus \operatorname{null}(T|_{\operatorname{range}(T-\lambda_1 \operatorname{Id}_V)} - \lambda_2 \operatorname{Id}_V) \oplus \operatorname{range}((T|_{\operatorname{range}(T-\lambda_1 \operatorname{Id}_V)} - \lambda_2 \operatorname{Id}_V)) \\ &= \cdots, \end{aligned}$$

   which eventually stabilizes to:

   $$V = E(\lambda_1, T) \oplus E(\lambda_2, T_1) \oplus \cdots E(\lambda_m, T_{m-1}), \tag{1}$$

   where T_i is the restriction of T to some T-invariant subspace. It is clear that $E(\lambda_i, T_{i-1}) \subseteq E(\lambda_i, T)$. We claim that $E(\lambda_i, T) \subseteq E(\lambda_i, T_{i-1})$, which is clear from (1).

   (1) ⇒ (3): let B_V be a basis of V such that $\mathcal{M}(T, B_V) = \operatorname{diag}(\lambda_1, \cdots, \lambda_k)$, where $\operatorname{diag}(\lambda_1, \cdots, \lambda_k)$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_k$. Notice that the matrix corresponding to $\prod_{i=1}^{m}(T - \lambda_i)$ is $\operatorname{diag}(\prod_{i=1}^{m}(\lambda_1 - \lambda_i), \cdots, \prod_{i=1}^{m}(\lambda_1 - \lambda_n))$ which is 0, because each of the products $\prod_{i=1}^{m}(\lambda_j - \lambda_i)$ vanish. Thus, $p_T(T) = 0$ is a multiple of the minimal polynomial. Since each of factors $(x - \lambda_i)$ appears only once, if the minimal polynomial q is not p_T, then there exists $j \in \{1, \ldots, k\}$ such that the j diagonal entry of $q(T)$ is non-zero, which implies that $q(T) \neq 0$, a contradiction.

   (3) ⇒ (1): this is done in the textbook, see the proof of 5.62.

   (3) ⇔ (4): is clear.

   (3) ⇒ (5): assume that p_T and p_T' have a common zero α. Since $p_T'(z) = \sum_{j=1}^{m} \prod_{i \neq j}(z - \lambda_i)$ we have:

   $$\sum_{j=1}^{m} \prod_{i \neq j}(\alpha - \lambda_i) = 0 \quad \prod_{i \neq j \quad \lambda_i \neq \alpha}(\alpha - \lambda_i) = 0,$$

however, the last equation is a contradiction.

(5) ⇒ (3): since we are over $\mathbb{C}$ we know that $p_T(z) = \prod_{i=1}^m (z - \lambda_i)$, where λ_i are the eigenvalues of T. Any zero of p_T is λ_j, but we notice that:

$$p'_T(\lambda_j) = \sum_{j=1}^{m} \prod_{i \neq j} (\alpha - \lambda_i) = \prod_{i \neq j} (\alpha - \lambda_i),$$

which is only non-zero if λ_j is not equal to any other λ_i.

(5) ⇒ (6): assume that q is a common factor, since we are over $\mathbb{Z}$ there exists $\alpha \in \mathbb{C}$ such that $(z-\alpha)$ is a common factor of p_T and p'_T, which implies that $p_T(\alpha) = p'_T(\alpha)$ which is a contradiction with (5).

(6) ⇒ (5): if α is such that $p_T(\alpha) = p'_T(\alpha)$, then Lemma 28 in the Lecture notes imply that $p_T = q_1(z-\alpha)$ and $p'_T = q_2(z-\alpha)$, for some $q_1, q_2 \in \mathbb{Z}[z]$, which contradicts (6).

2. Let V be a finite-dimensional vector space over $\mathbb{C}$. Let $\mathcal{E} \subseteq \mathcal{L}(V)$ be a subset of linear operators which commute, i.e. for any $S, T \in \mathcal{E}$ we have $ST = TS$.

   (i) Prove that for any $S, T \in \mathcal{E}$ the subspaces $\operatorname{null} p(S)$ and $\operatorname{range} p(S)$ are invariant under T.

   Solution. *First we notice that for any polynomial $p(z) \in \mathbb{C}[z]$ we have*

$$Tp(S) = T \sum_{i=0}^{m} a_i S^i = \sum_{i=0}^{m} a_i S^i T = p(S)T,$$

   since $TS^i = S^i T$ for every $i \geq 0$.
   Invariance of null $p(S)$: let $v \in$ null $p(S)$, i.e., $p(S)(v) = 0$. This follows from:

$$p(S)(T(v)) = T(p(S)(v)) = T(0) = 0.$$

   Invariance of range $p(S)$: let $v \in$ range $p(S)$, i.e., there exists a vector $u \in V$ such that $p(S)(u) = v$. We need to show that $T(v) \in$ range $p(S)$. Again we compute:

$$T(v) = T(p(S)(u)) = p(S)(T(u)),$$

   thus $T(v) = p(S)(T(u))$, i.e. $T(v) \in \operatorname{range} p(S)$.

   (ii) Prove that there is a vector in V that is an eigenvector for every element of $\mathcal{E}$.

   Solution. *Let $T \in \mathcal{E}$, consider λ an eigenvalue of T, which exists by Proposition 4 in the Lecture Notes, and $E(\lambda, T)$ the corresponding eigenspace. Let $S \in \mathcal{E}$ be any other element, by (i) $E(\lambda, T)$ is invariant under S, thus we obtain $S|_{E(\lambda,T)} : E(\lambda, T) \to E(\lambda, T)$ has an eigenvector $v \in E(\lambda, T)$ by Proposition 4 again. Notice that this implies that $T(v) = \lambda v$ and $S(v) = \mu S$ for some $\mu \in \mathbb{F}$. Let $\{T_1, T_2, \ldots\} = \mathcal{E}$ be an order on $\mathcal{E}$ by proceeding as before we obtain that there exist $v \in E(\lambda_i, T_i)$ for all $i \geq 1$ such that $T_i(v) = \lambda_i v$, i.e. v is an eigenvector of all $T \in \mathcal{E}$.*

   (iii) Prove that there is a basis of V with respect to which every element of $\mathcal{E}$ has an upper-triangular form.

   Solution. *We proceed by induction on $\dim V = n$. The case $n = 1$ is clear, since any 1 by 1 matrix is upper-triangular.*

Assume the result holds for all complex vector spaces of dimension $< n-1$. By (ii) let v_1 be a common eigenvector of all $T \in \mathcal{E}$, then $T(v_1) \in \operatorname{Span}\{v_1\}$ for every $T \in \mathcal{E}$. Let $W \subset V$ be a subspace such that $W \oplus \operatorname{Span}\{v_1\} = V$. Let $P : V \to W$ be the projection onto W. For every $T \in \mathcal{E}$, let:

$$\hat{T} := P \circ T|_W : W \to W.$$

Given any $S, T \in \mathcal{E}$ and $w \in W$ we calculate:

$$\hat{S}\hat{T}(w) = \hat{S}(Tw - av_1) = P(S(Tw - av_1)) = P(S(T(w))) \quad \text{for some } a \in \mathbb{F}.$$

Similarly, we have:

$$\hat{T}\hat{S}(w) = \hat{T}(Sw - bv_1) = P(S(Tw - bv_1)) = P(T(S(w))) \quad \text{for some } b \in \mathbb{F}.$$

That is $\hat{S}\hat{T} = \hat{T}\hat{S}$ for every $S, T \in \mathcal{E}$. By the inductive hypothesis there exists a basis $B_W = \{v_2, \ldots, v_n\}$ of W such that $\mathcal{M}(\hat{T}, B_W)$ is upper-triangular for every $T \in \mathcal{E}$. Thus, $\mathcal{M}(B_V, T)$ where $B_V := \{v_1, \ldots, v_n\}$ is upper-triangular.

3. Let V be a finite-dimensional vector space.

   (i) Let $T \in \mathcal{L}(V)$ be an invertible operator and $B_V = \{v_1, \ldots, v_n\}$ is a basis such that $\mathcal{M}(B_V, T)$ is upper triangular with $\lambda_1, \ldots, \lambda_n$ on the diagonal. Show that $\mathcal{M}(B_V, T^{-1})$ is also upper triangular with $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ on the diagonal.

   Solution. *Let $T(v_j) = \sum_{k=1}^n b_{kj} v_k$, then $b_{kj} = 0$ for $k < j$. Let $T^{-1}(v_i) = \sum_{j=1}^n a_{ji} v_j$. Since*

$$\begin{aligned} v_i = T(T^{-1}(v_i)) &= \sum_{j=1}^n a_{ji} T(v_j) \\ &= \sum_{j=1}^n a_{ji} \lambda_j v_j + \sum_{j=1}^n a_{ji} \sum_{k=j+1}^n b_{kj} v_k. \end{aligned}$$

   By considering $i = n$, we have

$$v_n = a_{nn} \lambda_n v_n + \sum_{j=1}^{n-1} a_{jn} \lambda_j v_j,$$

   which implies that $a_{jn} = 0$ for $1 \leq j < n$ and $a_{nn} = \lambda_n^{-1}$. By considering $i = n-1$ we obtain $a_{j,n-1} = 0$ for $1 \leq j < n-1$ and $a_{n-1,n-1} = b_{n-1,n-1}^{-1}$ and so on.

   (ii) Give an example of $T \in \mathcal{L}(V)$ and B_V such that $\mathcal{M}(T, B_V)$ contains only 0's in the diagonal but T is invertible.

   Solution. *Consider $T : \mathbb{C}^2 \to \mathbb{C}^2$ such that*

$$\mathcal{M}(T, B_V) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

   where B_V is the standard basis. We can directly check that $T^2 = \mathrm{Id}_{\mathbb{C}^2}$.

   (iii) Give an example of $T \in \mathcal{L}(V)$ and B_V such that $\mathcal{M}(T, B_V)$ contains only non-zero elements in the diagonal but T is not invertible.

   Solution. *Consider $T : \mathbb{C}^2 \to \mathbb{C}^2$ such that*

$$\mathcal{M}(T, B_V) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

   where B_V is the standard basis. Since $T((1,-1)) = (0,0)$, we have that T is not injective, hence not invertible.