Math 2102: Homework 2
Due on: Feb. 26, 2024 at 11:59 pm.

All assignments must be submitted via Moodle. Precise and adequate explanations should be given to each problem. Exercises marked with Extra might be more challenging or a digression, so they won't be graded.

1. Let $U \subseteq V$ be a subspace and denote by $\pi : V \to V/U$ the quotient map. Let $\pi^* \in \mathcal{L}((V/U)^*, V^*)$ denote the associated dual linear map.

   (i) Show that π^* is injective.

   Solution. *Let $\lambda \in (V/U)^*$ we have $\pi^*(\lambda) = \lambda \circ \pi \in V^*$. Assume that $\pi^*(\lambda) = 0$, that is for every $v \in V$ we have $\lambda \circ \pi(v) = 0$. Suppose that $\lambda \neq 0$, then there exists $x \in V/U$ such that $\lambda(x) \neq 0$. Notice $\pi : V \to V/U$ is surjective, i.e. $x = \tilde{x} + U$ for some $\tilde{x} \in V$ by definition; thus $\pi(\tilde{x}) = x$. This gives that $\pi^*(\lambda)(\tilde{x}) = \lambda(\pi(\tilde{x}) = \lambda(x) = 0$, which is a contradiction. Thus $\lambda = 0$, i.e. $\ker \pi^* = 0$, which by Lecture Notes Lemma 11, implies that π^* is injective.*

   Notice that in the case where V is finite-dimensional we proved in Lecture Notes Lemma 25 that π^* is injective if and only if π is surjective, which was proved in the paragraph above. So in this case the proof is simpler.

   (ii) Show that $\operatorname{range} \pi^* = U^0$.

   Solution. *Let $\lambda \in \operatorname{range} \pi^*$ then $\lambda = \pi^*(\mu)$ for some $\mu \in (V/U)^*$. Notice that if $u \in U$ then $\pi(u) = 0$ (here this means $0_{V/U}$), since μ is linear we have $\lambda(u) = \pi^*(\mu)(u) = \mu(\pi(u)) = 0$ for every $u \in U$, thus $\operatorname{range} \lambda \subseteq U^0$.*

   Now assume that $\lambda \in U^0$, thus we have $\lambda : V \to \mathbb{F}$ such that the restriction of λ to U vanishes. By Lemma 20, there exists an unique linear map $\sigma : V/U \to \mathbb{F}$, i.e. $\sigma \in (V/U)^*$, such that $\lambda = \sigma \circ \pi = \pi^*(\sigma)$. Thus, $U^0 \subseteq (V/U)^*$.

   (iii) Conclude that π^* is an isomorphism between $(V/U)^*$ and U^0.

   Solution. *By Corollary 8 (iv), we have that $\pi^* : (V/U)^* \to V^*$ induces:*

   $$\bar{\pi}^* : (V/U)^* / \operatorname{null} pi^* \to V^*$$

   with $(V/U)^* / \operatorname{null} pi^* \simeq \operatorname{range} \bar{\pi}^*$. By (i) we have $\operatorname{null} \pi^* = 0$, which implies that $\bar{\pi}^* = \pi^*$, by (ii) we have that $\operatorname{range} \pi^* = U^0$. Thus, we conclude that $(V/U)^* \simeq U^0$.

2. Consider $V = \mathbb{C}((x))$ the set of Laurent series, i.e. $a \in \mathbb{C}((x))$ is given by a series $a(x) = \sum_{i \in \mathbb{Z}} a_i x^i$, where $a_i \in \mathbb{C}$, and such that there exists $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n < N$.

   (i) Check that V is a vector space. Is the set $\{x^n\}_{n \in \mathbb{Z}}$ a basis of V?

   Solution. *Notice that as a set one has a bijection $\varphi : \mathbb{C}((x)) \xrightarrow{\simeq} \mathbb{C}^{\mathbb{Z}}$ given by*

   $$\varphi(\sum_{i \in \mathbb{Z}} a_i x^i) : \mathbb{Z} \to \mathbb{C}, \qquad \varphi(\sum_{i \in \mathbb{Z}} a_i x^i)(n) = a_n.$$

   The addition and scalar multiplication on $\mathbb{C}((x))$ are defined exactly as the addition and scalar multiplication on $\mathbb{C}^{\mathbb{Z}}$. This shows that $\mathbb{C}((x))$ is a vector space over $\mathbb{C}$. The set $\{x^n\}_{n \in \mathbb{Z}}$ is not a basis, since for example the element $\sum_{i \geq 0} x^i \in \mathbb{C}((x))$ can not be written as a finite *linear combination of the vectors in $\{x^n\}_{n \in \mathbb{Z}}$.*

(ii) Given any $g \in V$ consider the map $L_g : V \to \mathbb{C}$ given by

$$L_g(f) = \mathrm{Res}(gf) := \text{coefficient of } x^{-1} \text{ in } g(x)f(x),$$

Prove that L_g is a well-defined linear map.

Solution. *Let $g = \sum_{i\in\mathbb{Z}} c_i x^i$ and consider $f = \sum_{i\in\mathbb{Z}} a_i x^i$ in V. We have:*

$$gf = \sum_{i\in\mathbb{Z}}(\sum_{j\in\mathbb{Z}} c_{i-j}a_j)x^i,$$

where we notice that the sum $\sum_{j\in\mathbb{Z}}(j+1)c_{i-j}a_j$ is finite since for $j \ll 0$ we have $a_j = 0$ and for $j \gg 0$ we have that $c_{i-j} = 0$. Thus the expression gf is well-defined and

$$L_g(f) = \sum_{j\in\mathbb{Z}} c_{i-j}a_j.$$

Consider $f = \sum_{i\in\mathbb{Z}} a_i$ and $h = \sum_{i\in\mathbb{Z}} b_i$ in V and $d \in \mathbb{C}$ we have:

$$\begin{aligned} L_g(f+dh) &= \sum_{j\in\mathbb{Z}} c_{i-j}(a_j + db_{j+1}) \\ &= \sum_{j\in\mathbb{Z}} c_{i-j}a_j + d\sum_{j\in\mathbb{Z}} c_{i-j}b_j \\ &= L_g(f) + dL_g(h). \end{aligned}$$

(iii) Consider $\varphi : V \to V^*$ given by $\varphi(g) := L_g$. Prove that φ is an isomorphism.

Solution. *Consider $f = \sum_{i\in\mathbb{Z}} a_i$ and $h = \sum_{i\in\mathbb{Z}} b_i$ in V where $a_k \neq b_k$ for some $k \in \mathbb{Z}$. And assume that $L_f = L_h$. Then if we consider:*

$$L_f(x^{-k}) = L_h(x^{-k}) \Rightarrow a_k = b_k,$$

which is a contradiction. This proves that φ is injective.
Let $\lambda : V \to \mathbb{C}$ be a linear functional, define:

$$c_{-i-1} := \lambda(x^i) \text{ for every } i \in \mathbb{Z}.$$

Notice that for $i \gg 0$ we have $c_{-i-1} = 0$, since $\lambda(x^i) = 0$ for $i \gg 0$, otherwise $\lambda(\sum_{i\geq 0} x^i) = \sum_{i\geq 0} c_{-i}$ would not be well-defined. Thus, we can define $g := \sum_{i\in\mathbb{Z}} c_i x^i$. We claim that $L_g = \lambda$. Indeed, for any $x^k \in V$ we have:

$$L_g(x^k) = \mathrm{Res}(\sum_{i\in\mathbb{Z}} c_i x^{i+k}) = c_{-k-1} = \lambda(x^k).$$

Thus, for an arbitrary $f = \sum_{k\in\mathbb{Z}} a_k x^k$ we have

$$L_g(f) = \lambda(f).$$

This proves that φ is surjective.

(iv) Let $\mathbb{C}[[x]] \subset \mathbb{C}((x))$ be the subset of Taylor series, i.e. $a \in \mathbb{C}[[x]]$ if $a = \sum_{n\geq 0} a_n x^n$ for some $a_i \in \mathbb{C}$ and let $\mathbb{C}[x^{-1}] \subset \mathbb{C}((x))$ denote the subset of $a \in \mathbb{C}((x))$ such that $a = \sum_{n\leq 0} a_n x^n$, with $a_m = 0$ for $m \ll 0$. Prove that $\mathbb{C}[[x]]$ and $\mathbb{C}[x^{-1}]$ are subspaces.

(v) Determine the range of φ restricted to $\mathbb{C}[[x]]$ and $\mathbb{C}[x^{-1}]$.

Solution. *Let $\Psi : V^* \to V$ be defined by $\Psi(\lambda) = \sum_{i\in\mathbb{Z}} \lambda(x^{-i-1})x^i$, which is well-defined by the Solution to (iii). Notice that $\varphi \circ \psi = \mathrm{Id}_{V^*}$ and $\mathrm{Id}_V = \psi \circ \varphi$. Thus, $\lambda \in \varphi(\mathbb{C}[[x]])$ if $\psi(\lambda) \in \mathbb{C}[[x]]$, i.e.*

$$\lambda(x^i) = c_{-i-1} = 0 \text{ for } -i-1 \leq 0,\ i \geq -1,$$

i.e. $\varphi(\mathbb{C}[[x]]) = (x^{-1}\mathbb{C}[[x]])^0$ where $x^{-1}\mathbb{C}[[x]] \subset \mathbb{C}((x))$ is the subspace of Laurent series such that $a_i = 0$ for $i \leq -2$. Similarly, we have $\lambda \in \varphi(\mathbb{C}[x^{-1}])$ if $\psi(\lambda) \in \mathbb{C}[x^{-1}]$, i.e.

$$\lambda(x^i) = c_{-i-1} = 0 \text{ for } -i-1 \geq 0,\ i \leq -1,$$

i.e. $\varphi(\mathbb{C}[x]) = (x^-1\mathbb{C}[x^{-1}])^0$, where $x^-1\mathbb{C}[x^{-1}] \subset$ is the subspace of Laurent series such that $a_i = 0$ for $i \geq 0$.

(vi) (Extra) Conclude that $\mathbb{C}[[x]] \simeq \mathbb{C}[x]^*$ and that $\mathbb{C}[x]^* \simeq \mathbb{C}[[x]]$.

Solution. *Notice that (v) gives that $\mathbb{C}[[x]] \simeq (x^{-1}\mathbb{C}[[x]])^0$. By Lemma 23 (i) in the Lecture Notes we have:*

$$(x^{-1}\mathbb{C}[[x]])^0 = \operatorname{null} \imath^*,$$

where $\imath : x^{-1}\mathbb{C}[[x]] \to \mathbb{C}((x))$ is the canonical inclusion. Now we claim that $\operatorname{null} \imath^* \simeq \mathbb{C}[x]^*$. Indeed, consider $\zeta : \mathbb{C}[x]^* \to \mathbb{C}((x))^*$ given by

$$\zeta(\lambda)(x^i) := \begin{cases} \lambda(x^{-i-2}) \text{ if } i \leq -2, \\ 0 \text{ else.} \end{cases}$$

It is easy to check that $\zeta(\mathbb{C}[x]^*) = \operatorname{null} \imath^*$. The argument for $\mathbb{C}[x]^* \simeq \mathbb{C}[[x]]$ is similar, using the other result in (v).

3. Let $T : V \to W$ be a linear operator between real vector spaces. We define:

$$T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}, \qquad T(u+iv) := T(u) + iT(v).$$

(i) Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $T_{\mathbb{C}}$.

Solution. *For this we need to assume that $\lambda \in \mathbb{R}$.*
Let λ be an eigenvalue of T with eigenvector $v \in V$, that $T(v) = \lambda v$ and $v \neq 0$. Consider $T_{\mathbb{C}}(v+iv) = T(v) + iT(v) = \lambda v + i\lambda v = \lambda(v+iv)$. So $v + iv$ is an eigenvector of $T_{\mathbb{C}}$ with eigenvalue λ.
Now consider $v \in V_{\mathbb{C}}$ such that $T_{\mathbb{C}}(v) = \lambda v$. We write as $v = v_r + iv_i$ for $v_r, v_i \in V$. Then we have:

$$\lambda(v_r + iv_i) = T_{\mathbb{C}}(v_r + iv_i) = T(v_r) + iT(v_i).$$

The real part gives $\lambda v_r = T(v_r)$ and the imaginary part gives $\lambda v_i = T(v_i)$. Since either v_r or v_i is non-zero, otherwise v is zero, we have that either v_r or v_i is an eigenvector of T with eigenvalue λ.

(ii) Prove that λ is an eigenvalue of $T_{\mathbb{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.

Solution. *Now consider $v \in V_{\mathbb{C}}$ such that $T_{\mathbb{C}}(v) = \lambda v$. We write as $v = v_r + iv_i$ for $v_r, v_i \in V$. Let $\lambda = (a+ib)$ then we have:*

$$(a+ib)(v_r + iv_i) = (av_r - bv_i) + i(bv_r + av_i) = T(v_r) + iT(v_i).$$

By equating the real and imaginary part we have: $T(v_r) = (av_r - bv_i)$ *and* $T(v_i) = (bv_r + av_i)$. *Now consider* $v' := v_r - iv_i$, *then we have:*

$$\begin{aligned} T_{\mathbb{C}}(v_r - iv_i) &= T(v_r) - iT(v_i) \\ &= (av_r - bv_i) - i(bv_r + av_i) \\ &= (a - ib)(v_r - iv_i). \end{aligned}$$

Thus $v_r - iv_i$ *is an eigenvector with eigenvalue* $\overline{\lambda} = a - ib$. *The other direction has exactly the same proof.*

4. Let V be a finite-dimensional vector space and consider $T \in \mathcal{L}(V)$ and $U \subseteq V$ a subspace invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by:

$$T/U : V/U \to V/U, \qquad T/U(v + U) := T(v) + U.$$

(i) Check that T/U is well-defined.

Solution. *Let* $v + U = v' + U$ *in* V/U *we need to check that the assignments* $T/U(v + U) := T(v) + U$ *and* $T/U(v' + U) = T(v') + U$ *produce the same answer. Indeed, let* $u \in U$ *such that* $v - v' = u$, *then we have* $T(v) - T(v') = T(v - v') = T(u) \in U$ *since* U *is invariant under* T. *Thus,* $T(v) + U = T(v') + U$.

One also needs to check that T/U *is linear. Consider* $a \in \mathbb{F}$ *and* $u + U, u' + U \in V/U$ *then we have:*

$$T/U(a(u+U)+u'+U) = T(au+u')+U = aT(u)+T(u')+U = a(T/U(u)+U)+(T/U(u')+U).$$

(ii) Show that each eigenvalue of T/U is an eigenvalue of T.

Solution. *Let* λ *be an eigenvalue of* T/U, *then there exist* $x \in V/U$ *such that* $x \neq 0$ *and* $T/U(x) = \lambda x$. *Consider* $W \subseteq V$ *such that* $W \oplus U = V$, *which exists by Lecture Notes Lemma 5, since* V *is finite-dimensional.*

There exists an unique $w \in W$ *such that* $w + U = x$. *Indeed, assume that there are* $w, w' \in W$ *such that* $w + U = w' + U$, *then* $w - w' \in U$, *since* $U \cap W = \{0\}$, *we get* $w = w'$. *Now we compute:*

$$T(w) + U = T/U(w + U) = \lambda(w + U),$$

which implies that $T(w) - \lambda(w) \in U$.

Let $X := \operatorname{Span}(w) \oplus U$. *We just showed that* $(T - \lambda \operatorname{Id}_X)(X) \subseteq U \subseteq X$. *Since* $\dim U = \dim X - 1$ *we have that the linear map* $T|_X - \lambda \operatorname{Id}_X$ *is not surjective, since* X *is finite-dimensional* $T|_X - \lambda \operatorname{Id}_X$ *is also not injective. Thus, there exist* $v \in X \subseteq V$ *such that* $T(v) - \lambda v = 0$, *as we needed to prove.*

(iii) Prove that the minimal polynomial of T is a multiple of the minimal polynomial of T/U.

Solution. *Let* p *be the minimal polynomial of* T *on* V. *Notice that* U *is invariant under* $p(T)$ *and that* $p(T)/U = p(T/U)$. *Indeed, it is enough to check that*

$$T^n/U(v + U) = T^n(v) + U$$

for each $n \geq 1$. *Thus,* $p(T/U) = 0$, *since* $p(T) = 0$. *So by Lemma 33 we have that* p *is a multiple of the minimal polynomial of* T/U *as we needed to prove.*

(iv) Prove that $p_{T/U} p_{T|U}$ is a multiple of p_T, here p_S is the minimal polynomial of the operator S.

Solution. *Let r be the minimal polynomial of $T|_U : U \to U$. Let q be the minimal polynomial of T/U, then for every $v \in V$ we have*

$$q(T/U)(v+U) = q(T)(v) + U = U. \tag{1}$$

Then we claim that $r \circ q(T) = 0$. Indeed, let $v \in V$ then we have:

$$r(T) \circ q(T)(v) = r(T)(v) = 0$$

since by (1) $q(T)(v) \in U$ and $r(T) = 0$ on U.