

Math 2102: Homework 1

Due on: Feb. 5, 2024 at 11:59 pm.

All assignments must be submitted via Moodle. Precise and adequate explanations should be given to each problem. Exercises marked with Extra might be more challenging or a digression, so they won't be graded.

1. Let $T \in \mathcal{L}(V)$ and consider bases $B_1 = \{v_1, \dots, v_n\}$ and $B_2 = \{u_1, \dots, u_n\}$ of V . Prove that the following conditions are equivalent:

- (1) T is injective.
- (2) The columns of $\mathcal{M}(T, B_1, B_2)$ are linearly independent in \mathbb{F}^n .
- (3) The columns of $\mathcal{M}(T, B_1, B_2)$ span \mathbb{F}^n .
- (4) The rows of $\mathcal{M}(T, B_1, B_2)$ are linearly independent in \mathbb{F}^n .
- (5) The rows of $\mathcal{M}(T, B_1, B_2)$ span \mathbb{F}^n .

2. Let V be a finite-dimensional vector space and $\mathcal{L}(V)$ the space of linear maps from V to itself. Given a linear operator $T \in \mathcal{L}(V)$.

- (i) Assume that $TS = ST$, for every $S \in \mathcal{L}(V)$. Prove that $T = \lambda \text{Id}_V$ for some $\lambda \in \mathbb{F}$, where Id_V is the identity operator on V .
- (ii) Assume that $T = \lambda \text{Id}_V$ for some $\lambda \in \mathbb{F}$. Prove that T is represented by a diagonal matrix with entries $\lambda \in \mathbb{F}$ for *any basis* of V .
- (iii) Assume that T is invertible. Let B be a basis of V and $\mathcal{M}(T) := \mathcal{M}(T, B, B)$ the matrix representing T in the basis B . Prove that $\mathcal{M}(T)$ is invertible and that $\mathcal{M}(T)^{-1}$ represents the inverse of T in the basis B .

3. Recall the following construction from Example 3 in the lecture notes. Given V a vector space over \mathbb{R} . We define its complexification $V_{\mathbb{C}}$ as follows:

- as a set we let $V_{\mathbb{C}} := V \times V$;
- $+$: $V_{\mathbb{C}} \times V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ is given by $(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$;
- scalar multiplication is defined as $(a + bi) \cdot (u_1, v_1) = (au_1 - bv_1, bu_1 + av_1)$.

- (i) Check that $V_{\mathbb{C}}$ is a vector space over \mathbb{C} .
- (ii) Given W a vector space over \mathbb{C} we denote by \bar{W} the same set W seen as a vector space over \mathbb{R} . Explain why \bar{W} is a vector space over \mathbb{R} .
- (iii) Prove that $\iota : V \rightarrow V_{\mathbb{C}}$ given by $\iota(v) = (v, 0)$ is a linear map when both V and $V_{\mathbb{C}}$ are seen as vector spaces over \mathbb{R} . Argue why any linear map $\varphi : V \rightarrow W$ into a real vector space W can be extended as an \mathbb{R} -linear map from $V_{\mathbb{C}}$.
- (iv) Let V be a vector space over \mathbb{R} and W a vector space over \mathbb{C} . We will denote by $\mathcal{L}_{\mathbb{R}}(V, \bar{W})$ the set of linear maps between V and \bar{W} as vector spaces over \mathbb{R} and by $\mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ the set of linear maps between $V_{\mathbb{C}}$ and W as vector spaces over \mathbb{C} . Prove that there exists a bijective function:

$$\varphi : \mathcal{L}_{\mathbb{R}}(V, \bar{W}) \rightarrow \mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W).$$

(Hint: In how many ways can the map from (iii) be extended?)

- (v) (Extra) Is the function φ above linear? Notice that you first need to answer think about why the sets are vector spaces and over which field they are vector spaces.
- (vi) (Extra) Does φ preserve properties of the linear operators? For instance, if $T \in \mathcal{L}_{\mathbb{R}}(V, \bar{W})$ is injective is $\varphi(T)$ injective? Same question for surjective.
4. Let V and W be finite-dimensional vector spaces over a field \mathbb{F} .
- Given a subspace $U \subset V$, prove that there exists $T \in \mathcal{L}(V, W)$ with $\text{null } T = U$ if and only if $\dim U \geq \dim V - \dim W$.
 - Prove that $T \in \mathcal{L}(V, W)$ is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $ST = \text{Id}_V$, the identity operator on V .
 - Prove that $T \in \mathcal{L}(V, W)$ is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $TS = \text{Id}_W$, the identity operator on W .
5. Let V_1 and V_2 be two vector spaces over a field \mathbb{F} . Here is a definition: U is a *coproduct* of V_1 and V_2 if
- there are linear maps $\iota_1 : V_1 \rightarrow U$ and $\iota_2 : V_2 \rightarrow U$;
 - for any vector space W and linear maps $f_1 : V_1 \rightarrow W$ and $f_2 : V_2 \rightarrow W$, there exists an *unique* morphism $g : U \rightarrow W$ such that

$$g \circ \iota_1 = f_1 \text{ and } g \circ \iota_2 = f_2.$$

- Give examples of U satisfying condition (a) above.
 - Prove that there exists U satisfying (a) and (b) above.
 - Let U and U' be two vector spaces satisfying (a) and (b) above. Prove that U and U' are isomorphic.
6. Let V_1 and V_2 be two vector spaces over a field \mathbb{F} . A vector space Z is a *product* of V_1 and V_2 if
- there are linear maps $\pi_1 : Z \rightarrow V_1$ and $\pi_2 : Z \rightarrow V_2$;
 - for any vector space W and linear maps $f_1 : W \rightarrow V_1$ and $f_2 : W \rightarrow V_2$, there exists an *unique* morphism $h : W \rightarrow Z$ such that
- $$\pi_1 \circ h = f_1 \text{ and } \pi_2 \circ h = f_2.$$
- Prove that there exists Z satisfying (a) and (b) above.
 - Let U be a coproduct of V_1 and V_2 as defined in Problem 5 and Z be a product of V_1 and V_2 . Prove that U and Z are isomorphic.
7. (Extra) Redo Problem 5 and 6, but considering only sets and functions. Is it the case that the coproduct and product are isomorphic (i.e. bijective as sets) in that case?