Math 2102: Homework 1
Solutions

1. Let $T \in \mathcal{L}(V)$ and consider bases $B_1 = \{v_1, \dots, v_n\}$ and $B_2 = \{u_1, \dots, u_n\}$ of V. Prove that the following conditions are equivalent:

   (1) T is injective.

   (2) The columns of $\mathcal{M}(T, B_1, B_2)$ are linearly independent in $\mathbb{F}^n$.

   (3) The columns of $\mathcal{M}(T, B_1, B_2)$ span $\mathbb{F}^n$.

   (4) The rows of $\mathcal{M}(T, B_1, B_2)$ are linearly independent in $\mathbb{F}^n$.

   (5) The rows of $\mathcal{M}(T, B_1, B_2)$ span $\mathbb{F}^n$.

   Solution. *Notice that the columns of* $\mathcal{M}(T, B_1, B_2)$ *are given by* $w_i = \mathcal{M}(V, B_2)(Tv_i)$ *for* $1 \leq i \leq n$.

   (1) $\Rightarrow$ (2). *Assume by contradiction that there exists a non-zero linear combination:*

   $$a_1 w_1 + \cdots + a_n w_n = 0.$$

   By the linearity of $\mathcal{M}(V, B_2)(-)$ *and* T *we have*

   $$0 = \sum_{i=1}^{n} a_i \mathcal{M}(V, B_2)(Tv_i) = \mathcal{M}(V, B_2)(\sum_{i=1}^{n} a_i T v_i) = \mathcal{M}(V, B_2)(T(\sum_{i=1}^{n} a_i v_i)).$$

   Since $\{v_1, \dots, v_n\}$ *is a basis the non-zero linear combination* $\sum_{i=1}^{n} a_i v_i \neq 0$. *This gives that* $\operatorname{null} T \neq \{0\}$, *which is a contradiction with* T *being injective.*

   (2) $\Rightarrow$ (1). *Suppose* T *is not injective, i.e.* $a_1, \dots, a_n \in \mathbb{F}$ *such that* $a_1 w_1 + \cdots + a_n w_n \in \operatorname{null} T$. *By similar computations, we see that* $a_1 v_1 + \cdots + a_n v_n = 0$, *which is absurd as* B_2 *is assumed to be a basis.*

   (2) $\Rightarrow$ (3). *We need to prove that* $\{w_1, \dots, w_n\}$ *span* $\mathbb{F}^n$. *Since* $\dim \mathbb{F}^n = n$, *the result follows from Lemma 7 (2) from the Lectures notes.*

   (3) $\Rightarrow$ (2). *Since there are only* n *columns in total, they would span a space with dimension less than* n *if they are not linearly independent.*

   The implications (3) $\Leftrightarrow$ (5) *follow from Lemma 13.*

   (5) $\Leftrightarrow$ (4). *This could be proven similar to* (2) $\Leftrightarrow$ (3).

2. Let V be a finite-dimensional vector space and $\mathcal{L}(V)$ the space of linear maps from V to itself. Given a linear operator $T \in \mathcal{L}(V)$.

   (i) Assume that $TS = ST$, for every $S \in \mathcal{L}(V)$. Prove that $T = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{F}$, where Id_V is the identity operator on V.

   Solution. *Let's assume by contradiction that there exists a non-zero vector* $v \in V$ *such that* $0 \neq T(v) \neq \lambda v$ *for all* $\lambda \in \mathbb{F}$. *Then* $\{v, T(v)\}$ *is a linearly independent set, by Example 8 (iii) in the Lecture Notes. Let* $\{v, T(v), v_3, \dots, v_n\}$ *be an extension of* $\{v, T(v)\}$ *to a basis of* V, *which always exists by Corollary 2. (2).*
   Now we define $S : V \to V$ *by* $S(T(v)) = v$ *and* $S(v) = 0$ *and* $S(v_i) = v_i$ *for* $i \in \{3, \dots, n\}$. *Then we have a linear opeartor* S *such that*

   $$v = S \circ T(v) \neq T \circ S(v) = T(0) = 0,$$

   which is a contradiction.

(ii) Assume that $T = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{F}$. Prove that T is represented by a diagonal matrix with entries $\lambda \in \mathbb{F}$ for *any basis* of V.

Solution. *Let $B_V = \{v_1, \ldots, v_n\}$ be a basis. Assume by contradiction that there exist $i \neq j$ such that $a_{i,j} \neq 0$. This gives $T(v_i) = a_{i,j}v_j + \sum_{k \neq i} a_{i,k}v_k = \lambda v_i$. Thus, we have:*

$$a_{i,j}v_j + (a_{i,i} - \lambda)v_i + \sum_{k \neq j, k \neq i} a_{i,k}v_k = 0.$$

Since $\{v_1, \ldots, v_k\}$ are linearly independent we obtain that $a_{i,k} = 0$ for all $k \neq i$ and that $a_{i,i} = \lambda$.

(iii) Assume that T is invertible. Let B be a basis of V and $\mathcal{M}(T) := \mathcal{M}(T, B, B)$ the matrix representing T in the basis B. Prove that $\mathcal{M}(T)$ is invertible and that $\mathcal{M}(T)^{-1}$ represents the inverse of T in the basis B.

Solution. *By Lemma 12 in the lecture notes we have:*

$$\mathcal{M}(T, B_V) \circ \mathcal{M}(T^{-1}, B_V) = \mathcal{M}(T \circ T^{-1}, B_V) = \mathcal{M}(\operatorname{Id}_V, B_V). \tag{1}$$

By (ii) one has that $\mathcal{M}(\operatorname{Id}_V, B_V)$ is the diagonal matrix with coefficients 1. Thus, by multiplying (1) by $\mathcal{M}(T, B_V)^{-1}$ we obtain:

$$\mathcal{M}(T^{-1}, B_V) = \mathcal{M}(T, B_V)^{-1},$$

as required.

3. Recall the following construction from Example 3 in the lecture notes. Given V a vector space over $\mathbb{R}$. We define its complexification $V_\mathbb{C}$ as follows:

- as a set we let $V_\mathbb{C} := V \times V$;
- $+ : V_\mathbb{C} \times V_\mathbb{C} \to V_\mathbb{C}$ is given by $(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$;
- scalar multiplication is defined as $(a + bi) \cdot (u_1, v_1) = (au_1 - bv_1, bu_1 + av_1)$.

(i) Check that $V_\mathbb{C}$ is a vector space over $\mathbb{C}$.

Solution. *Consider $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in V_\mathbb{C}$ and $(a+ib), (c+id) \in \mathbb{C}$ we need to check the list of axioms in Definition 1.20 from the textbook.*

- $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2) = (v_1, v_2) + (u_1, u_2)$;
- $((u_1, u_2) + (v_1, v_2)) + (w_1, w_2) = (u_1 + v_1, u_2 + v_2) + (w_1, w_2) = (v_1 + u_1 + w_1, v_2 + u_2 + w_2) = (v_1, v_2) + (u_1 + w_1, u_2 + w_2) = (u_1, u_2) + ((v_1, v_2) + (w_1, w_2))$;
- $((a+ib)(c+id))(u_1, u_2) = (ac - bd + i(bc + ad))(u_1, u_2) = ((ac - bd)u_1 - (bc + ad)u_2, (ac - bd)u_2 + (bc + ad)u_1) = (a(cu_1 - du_2) - b(du_1 + cu_2), b(cu_1 - du_2) + a(du_1 + cu_2)) = (a + ib)(cu_1 - du_2, du_1 + cu_2) = (a + ib)(c + id)(u_1, u_2)$;
- $(u_1, u_2) + (0, 0) = (u_1 + 0, u_2 + 0) = (u_1, u_2)$, *where $0 \in V$ is the zero vector in V*;
- $(u_1, u_2) + (-u_1, -u_2) = (u_1 - u_1, u_2 - u_2) = (0, 0)$;
- $(1, 0)(u_1, u_2) = (u_1, u_2)$;
- $(a+ib)((u_1, u_2) + (v_1, v_2)) = (a+ib)(u_1 + v_1, u_2 + v_2) = (a(u_1 + v_1) - b(u_2 + v_2), a(u_2 + v_2) + b(u_1 + v_1)) = (au_1 - bu_2 + av_1 - bv_2), au_2 - bu_1 + av_2 - bv_1) = (a+ib)(u_1, u_2) + (a+ib)(v_1, v_2)$;
- $((a+ib) + (c+id))(u_1, u_2) = ((a+c) + i(b+d))(u_1, u_2) = ((a+c)u_1 - (b+d)u_2, (a+c)u_2 + (b+d)u_1) = (au_1 - bu_2 + cu_1 - du_2, au_2 + bu_1 + cu_2 + du_1) = (a+ib)(u_1, u_2) + (c+id)(u_1, u_2)$.

(ii) Given W a vector space over $\mathbb{C}$ we denote by $\bar{W}$ the same set W seen as a vector space over $\mathbb{R}$. Explain why $\bar{W}$ is a vector space over $\mathbb{R}$.

Solution. *The reason why $\bar{W}$ is a vector space over $\mathbb{R}$ is that we can restrict the action of $\mathbb{C}$ to $\mathbb{R}$ to obtain an action of $\mathbb{R}$. In details, $(-) \cdot_{\bar{W}} (-)$ is defined such that the following commutative diagram commutes:*

$$\begin{tikzcd}
\mathbb{R} \times \bar{W} \arrow[r, "(-)\cdot_{\bar{W}}(-)"] \arrow[d] & \bar{W} \arrow[d] \\
\mathbb{C} \times W \arrow[r, "(-)\cdot(-)"'] & W
\end{tikzcd}.$$

To check that $\bar{W}$ is a vector space over $\mathbb{R}$ we notice that the only axioms which are not immediate are the ones involving scalar multiplication. The axiom that says that scalar multiplication on a vector space is associative follows by noticing that the diagram below

$$\begin{tikzcd}
\mathbb{R} \times \mathbb{R} \times \bar{W} \arrow[r, "m_{\mathbb{R}} \times \mathrm{Id}_{\bar{W}}"] \arrow[d, "\mathrm{Id}_{\mathbb{R}} \times (-)\cdot_{\bar{W}}(-)"'] & \mathbb{R} \times \bar{W} \arrow[d, "(-)\cdot_{\bar{W}}(-)"] \\
\mathbb{R} \times \bar{W} \arrow[r, "(-)\cdot_{\bar{W}}(-)"'] & \bar{W}
\end{tikzcd}$$

commutes, because the diagram

$$\begin{tikzcd}
\mathbb{C} \times \mathbb{C} \times W \arrow[r, "m_{\mathbb{C}} \times \mathrm{Id}_{W}"] \arrow[d, "\mathrm{Id}_{\mathbb{C}} \times (-)\cdot_{W}(-)"'] & \mathbb{C} \times W \arrow[d, "(-)\cdot_{W}(-)"] \\
\mathbb{C} \times W \arrow[r, "(-)\cdot_{W}(-)"'] & W
\end{tikzcd}$$

commutes, since W is a vector space. A similar argument proves the distributivity properties of scalar multiplications.

(iii) Prove that $\imath : V \to V_{\mathbb{C}}$ given by $\imath(v) = (v, 0)$ is a linear map when both V and $V_{\mathbb{C}}$ are seem as vector spaces over $\mathbb{R}$. Argue why any linear map $\varphi : V \to W$ into a real vector space W can be extended as an $\mathbb{R}$-linear map to $V_{\mathbb{C}}$.

Solution. *Let $v, u \in V$ and $a \in \mathbb{R}$ we calculate:*

$$\imath(av + u) = (av + u, 0) = (av, 0) + (u, 0) = a(v, 0) + (u, 0) = a\imath(v) + \imath(u).$$

This shows that $\imath$ is $\mathbb{R}$-linear, i.e. a linear map between real vector spaces. Let $\varphi : V \to W$ be a linear map between real vector spaces, we can consider $\tilde{\varphi} : V_{\mathbb{C}} \to W$ given by

$$\tilde{\varphi}(v_1, v_2) := \varphi(v_1).$$

We check this map is linear, consider $(v_1, v_2), (u_1, u_2) \in V_{\mathbb{C}}$ and $a \in \mathbb{R}$, then we have:

$$\tilde{\varphi}(a(v_1, v_2) + (u_1, u_2)) = \varphi(av_1 + u_1) = a\varphi(v_1) + \varphi(u_1) = a\tilde{\varphi}(v_1, v_2) + \tilde{\varphi}(u_1, u_2).$$

Notice that $\tilde{\varphi}$ is not $\mathbb{C}$-linear, i.e.

$$i\tilde{\varphi}(v_1, v_2) \neq \tilde{\varphi}(i(v_1, v_2)),$$

the left-hand side gives $i\varphi(v_1)$, whereas the right-hand side gives $-\varphi(v_2)$.

(iv) Let V be a vector space over $\mathbb{R}$ and W a vector space over $\mathbb{C}$. We will denote by $\mathcal{L}_{\mathbb{R}}(V, \bar{W})$ the set of linear maps between V and $\bar{W}$ as vector spaces over $\mathbb{R}$ and by $\mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ the set of linear maps between $V_{\mathbb{C}}$ and W as vector spaces over $\mathbb{C}$. Prove that there exists a bijective function:

$$\varphi : \mathcal{L}_{\mathbb{R}}(V, \bar{W}) \to \mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W).$$

(Hint: In how many ways can the map from (iii) be extended?)

Solution. *The comment at the end of solution for the previous item hints at how to write the function φ. Given $L \in \mathcal{L}_{\mathbb{R}}(V, \bar{W})$, we let*

$$\varphi(L)(v_1, v_2) := L(v_1) + iL(v_2).$$

We first check that $\varphi(L)$ is $\mathbb{C}$-linear. Given $(v_1, v_2), (u_1, u_2) \in V_{\mathbb{C}}$ and $(a + ib) \in \mathbb{C}$ we have:

$$\begin{aligned}
\varphi(L)((a + ib)(v_1, v_2) + (u_1, u_2)) &= \varphi(L)(av_1 - bv_1 + u_1, bv_1 + av_2 + u_2) \\
&= L(av_1 - bv_2 + u_1) + iL(bv_1 + av_2 + u_2) \\
&= aL(v_1) - bL(v_2) + L(u_1) + ibL(v_1) + iaL(v_2) + iL(u_2) \\
&= aL(v_1) + ibL(v_1) + (a + ib)iL(v_2) + L(u_1) + iL(u_2) \\
&= (a + ib)\varphi(L)(v_1, v_2) + \varphi(u_1, u_2).
\end{aligned}$$

Finally, we check that φ is a bijection. Consider the function $\psi : \mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W) \to \mathcal{L}_{\mathbb{R}}(V, \bar{W})$ given by:

$$\psi(S)(v) = S(v, 0).$$

We claim that ψ is an inverse to φ. Let's calculate, given $S \in \mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ for every $(v_1, v_2) \in V_{\mathbb{C}}$ we have:

$$\begin{aligned}
\varphi(\psi(S))(v_1, v_2) &= \psi(S)(v_1) + i\psi(S)(v_2) \\
&= S(v_1, 0) + iS(v_2, 0) \\
&= S((v_1, 0) + i(v_2, 0)) \\
&= S(v_1, v_2).
\end{aligned}$$

Thus, $\varphi \circ \psi = \mathrm{Id}_{\mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W)}$.
Similarly, for any $L \in \mathcal{L}_{\mathbb{R}}(V, \bar{W})$ and $v \in V$ we have:

$$\begin{aligned}
\psi(\varphi(L))(v) &= \varphi(L)(v, 0) \\
&= L(v) + iL(0) \\
&= L(v).
\end{aligned}$$

Thus, $\psi \circ \varphi = \mathrm{Id}_{\mathcal{L}_{\mathbb{R}}(V, \bar{W})}$.

(v) (Extra) Is the function φ above linear? Notice that you first need to think about why the sets are vector spaces and over which field they are vector spaces.

Solution. *Addition and scalar multiplication of linear maps could be done pointwise, and the resulting map is again linear. One should check the operations indeed define a vector space structure by going through the axioms.*
The base field of $\mathcal{L}_{\mathbb{R}}(V, \overline{W})$ is naturally $\mathbb{R}$, while that of $\mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ is naturally $\mathbb{C}$. However, the linearity of a map between vector spaces over different fields is not well-defined, so we have two choices: Realize $\mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ as an $\mathbb{R}$-vector space or define a $\mathbb{C}$-vector space structure on $\mathcal{L}_{\mathbb{R}}(V, \overline{W})$.

If $\mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ is realized as an $\mathbb{R}$-vector space, i.e. we only allow multiplication by scalars in $\mathbb{R}$. We compute

$$\begin{aligned}\varphi(aL_1 + L_2)(v_1, v_2) &= (aL_1 + L_2)(v_1) + i(aL_1 + L_2)(v_2)\\ &= aL_1(v_1) + iaL_1(v_2) + L_2(v_2) + iL_2(v_2)\\ &= a\varphi(L_1)(v_1, v_2) + \varphi(L_2)(v_1, v_2)\end{aligned}$$

where $a \in \mathbb{R}$ and $(v_1, v_2) \in V_{\mathbb{C}}$. Thus, φ is $\mathbb{R}$-linear.
To define a $\mathbb{C}$-vector space structure, the rest is to define a scalar multiplication of complex numbers, which could be easily done as the $\mathbb{C}$-vector space structure on W allows us to use the pointwise multiplication. By similar computations above, we may conclude that φ is $\mathbb{C}$-linear in this case.

(vi) (Extra) Does φ preserve properties of the linear operators? For instance, if $T \in \mathcal{L}_{\mathbb{R}}(V, \bar{W})$ is injective is $\varphi(T)$ injective? Same question for surjective.

Solution. *Suppose $T \in \mathcal{L}_{\mathbb{R}}(V, \bar{W})$ such that $\ker T$ is trivial. If $\varphi(T)(u, v) = 0$, then $Lu + iLv = 0$, i.e. $Lu = Lv = 0$, and hence $u = v = 0$, which implies $(u, v) = 0$, i.e. $\varphi(L)$ is still injective. [Stop and think about the meaning of 0's here.]*
Suppose $T \in \mathcal{L}_{\mathbb{R}}(V, \bar{W})$ is surjective, i.e. $\forall\, w \in W$, $\exists\, v \in V$ such that $Tv = w$, which implies $\varphi(T)(v, 0) = w$, so $\varphi(T)$ is still surjective.

4. Let V and W be finite-dimensional vector spaces over a field $\mathbb{F}$.

(i) Given a subspace $U \subset V$, prove that there exists $T \in \mathcal{L}(V, W)$ with $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution. *Assume that there exist U such that $\operatorname{null} T = U$. Then the fundamental theorem of linear algebra gives:*

$$\dim U = \dim \operatorname{null} T = \dim V - \dim \operatorname{range} T.$$

Since $\operatorname{range} T \subseteq W$, we have $\dim W \geq \dim \operatorname{range} T$, which gives that

$$\dim V - \dim W \leq \dim V - \dim \operatorname{range} T = \dim U.$$

Now assume that $\dim V - \dim W > \dim U$ this gives that:

$$\dim V - \dim \operatorname{range} T \geq \dim U \Rightarrow \dim \operatorname{null} T > \dim U,$$

which implies that $\operatorname{null} T \neq U$.

(ii) Prove that $T \in \mathcal{L}(V, W)$ is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $ST = \mathrm{Id}_V$, the identity operator on V.

Solution. *Assume that there exist S such that $ST = \mathrm{Id}_V$. Assume by contradiction that T is not injective, then there exist $v_1, v_2 \in V$ such that $v_1 \neq v_2$ and $T(v_1) = T(v_2)$. But applying S we obtain that $v_1 = ST(v_1) = ST(v_2) = v_2$, which is a contradiction.*
Assume that T is injective. Consider the subspace $\operatorname{range} T \subseteq W$, we can find $U \subseteq W$ such that $U \oplus \operatorname{range} T = W$, since W is finite-dimensional (this is 2.33 in the textbook). Now we define $S : W \to V$ as follows:

$$S(w) = \begin{cases} v \text{ if } w \in \operatorname{range} T \\ 0 \text{ else} \end{cases}.$$

Notice that this is well-defined, because T is injective and we decomposed W as a direct sum, so each w can be uniquely written as $w_1 + w_2$, where $w_1 \in \operatorname{range} T$ and $w_2 \in U$. We now notice that $ST(v) = v$ for every $v \in V$.

(iii) Prove that $T \in \mathcal{L}(V, W)$ is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $TS = \mathrm{Id}_W$, the identity operator on W.

Solution. *Assume that $TS = \mathrm{Id}_W$. Let $w \in W$, then we have that $S(w) \in V$ and $T(S(w)) = w$, thus T is surjective.*

Assume that T is surjective and consider $\operatorname{null} T \subseteq V$. Because V is finite-dimensional, again 2.33 in the textbook, guarantees that there exist $U \subseteq V$ such that $\operatorname{null} T \oplus U = V$. We define:

$$S(w) = \begin{cases} v \text{ if } v \in U \text{ and } T(v) = w \\ 0 \text{ else} \end{cases} .$$

We claim that S is well-defined. Indeed, suppose that we have $v_1, v_2 \in U$ such that $T(v_1) = T(v_2)$ then $v_1 - v_2 \in \operatorname{null} T$ and $v_1 - v_2 \in U$. Since $\operatorname{null} T \cap U = \{0\}$ we have that $v_1 = v_2$. Now notice that for any $w \in W$ we have $TS(w) = T(v) = w$, by definition.

5. Let V_1 and V_2 be two vector spaces over a field $\mathbb{F}$. Here is a definition U is a *coproduct* of V_1 and V_2 if

(a) there are linear maps $\imath_1 : V_1 \to U$ and $\imath_2 : V_2 \to U$;

(b) for any vector space W and linear maps $f_1 : V_1 \to W$ and $f_2 : V_2 \to W$, there exists an *unique* morphism $g : U \to W$ such that

$$g \circ \imath_1 = f_1 \text{ and } g \circ \imath_2 = f_2.$$

(i) Give examples of U satisfying condition (a) above.

Solution. *We can take $U = \{0\}$ and $\imath_1 = \imath_2$ to be the zero map. I guess it turns out that requiring just condition (i) is not that meaningful.*

(ii) Prove that there exists U satisfying (a) and (b) above.

Solution. *Let $U = V_1 \times V_2$ and $\imath_1(v_1) := (v_1, 0)$ and $\imath_2(v_2) = (0, v_2)$.*
We check it satisfies the required property. We define $h : V_1 \times V_2 \to W$ as follows:

$$g(v_1, v_2) := f_1(v_1) + f_2(v_2).$$

It is linear as follows from:

$$\begin{aligned} g(a(v_1, v_2) + (u_1, u_2)) &= f_1(av_1 + u_1) + f_2(av_2 + u_2) \\ &= af_1(v_1) + af(v_2) + f_1(u_1) + f_2(u_2) \\ &= ah(v_1, v_2) + h(u_1, u_2) \end{aligned}$$

for any $a \in \mathbb{F}$ and $v_1, u_1 \in V_1$ and $v_2, u_2 \in V_2$.
Now notice that $g \circ \imath_1(v) = g(v, 0) = f_1(v)$ for $v \in V_1$ and $g \circ \imath_2(u) = g(0, u) = f_2(u)$ for $u \in V_2$, as required.
Finally, we need to check that g is unique satisfying the above equations. Assume there exists $h : V_1 \times V_2 \to W$ such that $h \circ \imath_i = f_i$ for $i = 1, 2$. Notice that for arbitrary $(v_1, v_2) \in V_1 \times V_2$ we have:

$$\begin{aligned} (g - h)(v_1, v_2) &= (g - h)((v_1, 0)) + (g - h)((0, v_2)) \\ &= (g - h) \circ \imath_1(v_1) + (g - h)\imath_2(v_2) = 0, \end{aligned}$$

where we used that $g-h$ is still a linear map, since the set $\mathcal{L}(V_1\times V_2, W)$ is a vector space. Thus, we get $g-h=0 \Rightarrow g=h$.

(iii) Let U and U' be two vector spaces satisfying (a) and (b) above. Prove that U and U' are isomorphic.

Solution. *Assume that there exists U' and morphisms $\imath'_i : V_i \to U'$ such that for every two linear maps $h_i : V_i \to W$ there exists an* unique *$g' : U' \to W$ such that $g' \circ \imath'_i = h_i$.*
Consider the diagram:

$$\begin{tikzcd}
V_1 \arrow[d, "\imath_1"'] \arrow[dr, "\imath'_1"] \arrow[drr, "\imath_1"] & & \\
V_1 \times V_2 \arrow[r, "g"] & U' \arrow[r, "g'"] & V_1 \times V_2 \\
V_2 \arrow[u, "\imath_2"] \arrow[ur, "\imath'_2"] \arrow[urr, "\imath_2"'] & &
\end{tikzcd}.$$

We explain: g exists because $V_1\times V_2$ is a coproduct, and g' exists because U' is a coproduct. Now in the diagram above we notice that because $V_1\times V_2$ is a coproduct, there exist an unique *linear map $k : V_1\times V_2 \to V_1\times V_2$ such that $k\circ\imath_1 = \imath_1$ and $k\circ\imath_2=\imath_2$. Clearly, $k=\mathrm{Id}_{V_1\times V_2}$ satisfies this property, thus $g'\circ g$ which also satisfies this property because the diagram above commutes has to be equal to $\mathrm{Id}_{V_1\times V_2}$. The same reasoning applied to the diagram:*

$$\begin{tikzcd}
V_1 \arrow[d, "\imath'_1"'] \arrow[dr, "\imath_1"] \arrow[drr, "\imath'_1"] & & \\
U' \arrow[r, "g'"] & V_1 \times V_2 \arrow[r, "g"] & U' \\
V_2 \arrow[u, "\imath'_2"] \arrow[ur, "\imath_2"] \arrow[urr, "\imath'_2"'] & &
\end{tikzcd}.$$

shows that $g\circ g' = \mathrm{Id}_{U'}$.

6. Let V_1 and V_2 be two vector spaces. A vector space Z is a *product* of V_1 and V_2 if

(a) there are linear maps $\pi_1 : Z \to V_1$ and $\pi_2 : Z \to V_2$;

(b) for any vector space W and linear maps $f_1 : W \to V_1$ and $f_2 : W \to V_2$, there exists an *unique* morphism $h : W \to Z$ such that

$$\pi_1 \circ h = f_1 \text{ and } \pi_2 \circ h = f_2.$$

(i) Prove that there exists Z satisfying (a) and (b) above.

Solution. *We claim that $Z = V_1\times V_2$ with $\pi_1(v_1,v_2)=v_1$ and $\pi_2(v_1,v_2)=v_2$ is a valid construction of the product in $\mathbb{F}$-*Vect*. Let $f_1 : W\to V_1, f_2 : W \to V_2$ be arbitrary linear maps. Assume such h exists, we may conclude $h(w) = (f_1(w), f_2(w))$ from $\pi_1\circ h(w) = f_1(w)$ and*

$\pi_2 \circ h(w) = f_2(w)$ *by definition of* π_1, π_2*, so the uniqueness is guaranteed. We should check the* h *we defined is a* $\mathbb{F}$*-linear map (*$\mathbb{F}$-Vect *morphism) to prove the existence. Notice*

$$\begin{aligned} h(aw + w') &= (f_1(aw + w'), f_2(aw + w')) \\ &= (af_1(w) + f_1(w'), af_2(w) + f_2(w')) \\ &= a(f_1(w), f_2(w)) + (f_1(w'), f_2(w')) \\ &= ah(w) + h(w') \end{aligned}$$

for any $a \in \mathbb{F}$ *and* $w, w' \in W$*, so we are done.*

(ii) Let U be a coproduct of V_1 and V_2 as defined in Problem 5 and Z be a product of V_1 and V_2. Prove that U and Z are isomorphic.

Solution. *Similar to the proof in 5(iii), we may prove that the construction by universal properties is always unique up to isomorphisms, so it suffices to show that one of the products we constructed is isomorphic to one of the coproducts we constructed. However, this is trivial as we are using the exact same* $\mathbb{F}$*-vector space as the product and coproduct.*

7. (Extra) Redo Problem 5 and 6, but considering only sets and functions. Is it the case that the coproduct and product are isomorphic (i.e. bijective as sets) in that case?

   Solution. *One may easily check the product in* Set *is given by the Cartesian product, while the coproduct is given by the disjoint union. In general, they are not isomorphic. For example,* $\{1,2\} \times \{3\} = \{(1,3),(2,3)\}$ *contains two elements, while* $\{1,2\} \sqcup \{3\} = \{1,2,3\}$ *contains three elements.*