Derived Algebraic Geometry Seminar

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These notes are for a seminar in DAG at the Chinese University of Hong Kong in the Fall 2021.
I will try to edit them locally using VS Code, integrated with Github to manage versions.
Our main goal will be to discuss must of Lurie's thesis [26].
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Chapter 1

Introduction

1.1 What is derived algebraic geometry?

1.1.1 Classical algebraic geometry

If we approach algebraic geometry from Grothendieck's functor of points perspective we can start developing the theory by considering the category of (classical 0-truncated) prestacks

$$\mathsf{PreStk} := \mathrm{Fun}(\mathsf{CAlg}_k, \mathrm{Sets})$$

i.e. the category of functors from the category of commutative k-algebras to sets.

Then we notice that the category $\mathsf{Sch}^{\mathrm{aff}} := \mathsf{CAlg}_k^{\mathrm{op}}$ admits many topologies: Zariski, étale, flat, etc. One can then restrict to the subcategory[1]

$$\mathsf{Stk} \hookrightarrow \mathsf{PreStk}$$

of these functors which are sheaves for the given topology[2], say we will stick with étale for this discussion.

One can go further and try to understand the stacks $X \in \mathsf{Stk}$ which are constructed from affine schemes by glueing. This possibly leads to a definition as follows:

Definition 1.1.1. An object $Z \in \mathsf{PreStk}$ is a *scheme*[3] if it satisfies:

1) Z is a sheaf for the étale topology;
2) the diagonal map $Z \to Z \times Z$ is (affine) representable, i.e. given any map $S \to Z \times Z$ for $S \in \mathsf{Sch}^{\mathrm{aff}}$ the fiber product
$$Z \underset{Z \times Z}{\times} S$$
is an affine scheme;
3) Z has a *Zariski cover* $p : U \to Z$, i.e.
$$U \simeq \mathrm{L}(\sqcup_I S_i),$$
where each $S_i \in \mathsf{Sch}^{\mathrm{aff}}$, the maps $p_i := p|_{S_i}$ are open embeddings[4] and for any affine scheme $T \in \mathsf{Sch}^{\mathrm{aff}}$ the induced map of affine schemes
$$p : U \underset{Z}{\times} T \to T$$

[1] This makes Stj a *localization* of PreStk since this inclusion admits a left adjoint, the sheafification functor.

[2] Recall this means that for $S' \to S$ a Zariski, étale, or flat covering the canonical map
$$F(S) \to \lim_{[n] \in \Delta^{\mathrm{op}}} F(S' \underset{S}{\times} \cdots \underset{S}{\times} S')$$
where we take the product of $(n+1)$ copies of S' over S. In fact, we only need to consider the limit over the subcategory $\Delta^{\leq 2, \mathrm{op}}$, i.e. the full subcategory of Δ^{op} generated by the objects $[0], [1]$ and $[2]$.

[3] Technically these are separated and quasi-compact, but we won't worry about that here.

[4] We make sense of this condition by using (ii). Indeed, a representable map $f : U \to Z$ is an open embedding if for any $S \to Z$, where $S \in \mathsf{Sch}$ the induced map
$$U \underset{Z}{\times} S \to S$$
is an open embedding of affine schemes.

is surjective. We also impose that I is finite.

From the above definition it is clear that the category of schemes Sch forms a fully faithful subcategory of Stk and in turn of PreStk as well. It can be checked that this definition agrees with the subcategory of locally ringed spaces which are locally isomorphic to spectra of commutative rings.

1.1.2 Derived algebraic geometry

The theory of derived geometry takes the ambitious (or maybe natural) stand that we should generalize the two categories involved in the definition of a prestack above. The category of commutative algebras over k should be replaced by the ∞-category CAlg_k of *derived algebras over* k. There are different ways to define this category, we can take either one of the following approaches[5]:

- CAlg_k is the ∞-category of simplicial commutative k algebras;
- CAlg_k is the ∞-category of connective commutative differential graded algebras (cdgas), i.e. complexes of vector spaces $A^\bullet$ endowed with a commutative multiplication and such that $H^k(A^\bullet) = 0$ for $k > 0$[6].
- CAlg_k is the ∞-category of $\mathbb{E}_\infty$-algebra[7] in the derived ∞-category of connective complexes of k-vector spaces.

Crucial to the understanding and development of derived algebraic geometry is the theory of ∞-categories. We will have an introduction to this theory in Chapter 2 for now the reader should think of an ∞-category $\mathscr{C}$ as a gadget that organizes objects and such that morphisms between any two objects form a topological space, which is *meaningful only up to homotopy*[8]. The prototypical ∞-category is that of spaces Spc, i.e. topological space considered up to homotopy. In particular, in Spc the object pt is equivalent (in the sense of category theory) to any contractible topological space.

Usual categories give examples of ∞-categories by considering every mapping set with the discrete topology, for instance, one has a fully faithful embedding $\mathrm{Sets} \hookrightarrow \mathrm{Spc}$ of the category of sets thought of as an ∞-category into the ∞-category of spaces.

Notice that the category Spc has many more subcategories, for any k let $\mathrm{Spc}^{\leq k} \hookrightarrow \mathrm{Spc}$ denote the full subcategory generated by spaces whose homotopy groups vanish for $i > k$. For $k = 0, 1$ we have equivalences $\mathrm{Sets} \simeq \mathrm{Spc}^{\leq 0}$ and $\mathrm{Grpd} \simeq \mathrm{Spc}^{\leq 1}$, where Grpd denotes the (1-)category of groupoids. The inclusion $\mathrm{Spc}^{\leq k} \hookrightarrow \mathrm{Spc}$ admits a left adjoint $\mathrm{Spc} \to \mathrm{Spc}^{\leq k}$ given by collapsing the cells of dimension bigger than k.

Similarly, the ∞-category of derived k-algebras CAlg_k has a subcategory $\mathrm{CAlg}_k^{\geq -n}$ whose objects are represented by complexes of vector spaces $A^\bullet$ such that $H^{-\ell} A^\bullet = 0$ for $\ell > n$. And we also have truncations $\mathrm{CAlg}_k \to \mathrm{CAlg}_k^{\geq -n}$ given by discarding the left-tails of the cdga[9].

[5]For k a field containing $\mathbb{Q}$ they are all equivalent, but they are not equivalent in general (see §4.1 for more details.)

[6]In these notes we will adopt the *cohomological* grading convention.

[7]Also called commutative algebra objects, in an ∞-category the only notion of a commutative algebra object is that of a homotopy coherent multiplication.

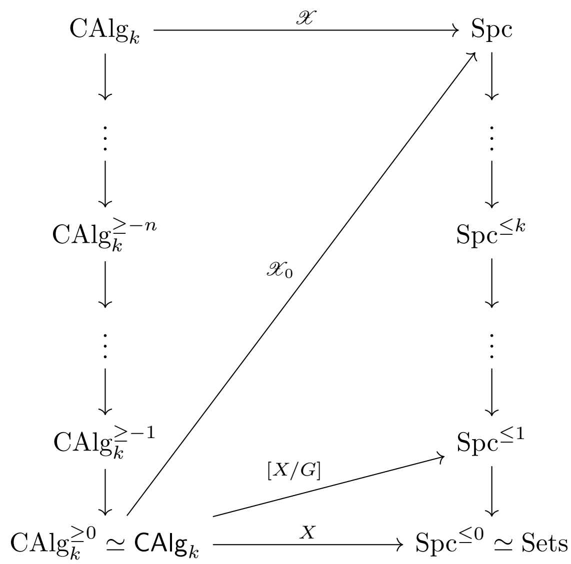
[8]In particular, the underlying set of the space of morphism doesn't have an intrinsic significance, e.g. a point and any contractible space are equivalent; however a point and two points are different since they have a different set of *connected components*.

[9]Concretely, given a cochain complex $A^\bullet$ one has

$$\tau^{\geq -n}(A^\bullet) = \cdots \to 0 \to \frac{A^{-n}}{\mathrm{Im}(d_{-n-1})} \to A^{-n+1} \to \cdots,$$

where $d_{-n-1} : A^{-n-1} \to A^{-n}$.

The story of derived algebraic geometry can synthesized in the following diagram



(1.1)

In the diagram (1.1) the data of:

- X is an example of algebraic space and if it satisfies (i-iii) above then it is a classical scheme;
- $[X/G]$ is an example an algebraic stack (in the sense of Deligne–Mumford or Artin);
- $\mathscr{X}_0$ is an example of an ∞-stack (in the sense of Simpson);
- $\mathscr{X}$ is an example of a derived stack.

As in usual algebraic geometry, derived schemes will correspond to a class of $\mathscr{X}$ that satisfy the natural analogues of Definition 1.1.1. By definition one has

$$\mathrm{Sch} \hookrightarrow \mathrm{Stk} \hookrightarrow \mathrm{PreStk}$$

where Sch is the ∞-category of derived schemes and Stk the ∞-category of prestacks satisfying the sheaf condition for the étale topology.

We notice that the data of $X : \mathrm{CAlg} \to \mathrm{Spc}$ is, in general, more than the data of the functor $X|_{\mathsf{CAlg}} : \mathsf{CAlg} \to \mathrm{Spc}$. In particular, if one has a classical scheme $Z_0 : \mathsf{CAlg} \to \mathrm{Sets}$ the data of a derived scheme $X : \mathrm{CAlg} \to \mathrm{Spc}$ such that

$$X|_{\mathsf{CAlg}} \simeq Z$$

is called a *derived enhacement of* Z. In particular, this suggests that the notion of derived scheme X should have the property that when restricted to a functor $X|_{\mathsf{CAlg}} : \mathsf{CAlg} \to \mathrm{Spc}$ is factors through the category $\mathrm{Sets} \hookrightarrow \mathrm{Spc}$.

Why do we care? This subject has long pre-history, a big circle of ideas that was present and that motivated lots of people to develop its foundations. Instead of listing any of that here we give *a posterior* justification by listing the applications of derived algebraic geometry to different areas of mathematics.

1.2 Formal properties

1.2.1 Tools for derived algebraic geometry

There are essentially two main tools that we will rely on in this introduction. The first is some homotopy theory, which is packaged in the theory of ∞-categories, stable ∞-categories and making sense of basic notions from commutative algebra in this context. The second is the cotangent complex.

Informally speaking, for almost any prestack $\mathscr{X} \in \text{PreStk}$ one can associated an object $T^*\mathscr{X} \in \text{QCoh}(\mathscr{X})$ of the ∞-category of quasi-coherent sheaves on $\mathscr{X}$. For the moment the reader can keep in mind that $\text{QCoh}(\text{pt}) \simeq \text{Vect}$ the derived ∞-category of cochain complexes over k, which is a derived enhancement of the usual derived category of k, i.e. $\text{h}\,\text{Vect} \simeq \mathsf{D}(k)$[10].

The object $T^*\mathscr{X}$ controls the deformation theory of $\mathscr{X}$, in other words–$T^*\mathscr{X}$ provides a linear invariant, i.e. a quasi-coherent sheaf, that deals with the question of how to extend a map $S \to \mathscr{X}$ to $S \hookrightarrow \tilde{S}$ a square-zero extension, i.e. $\tilde{S} \simeq \text{Spec}(\mathscr{O}_S \oplus \mathscr{F})$ for some quasi-coherent sheaf $\mathscr{F} \in \text{QCoh}(S)^{\leq 0}$.

We will define the cotangent complex in §7. For now we just list a couple of properties mention one property: for Z a derived scheme we will have $T^*Z \in \text{QCoh}(Z)^{\leq 0}$, i.e. its cotangent complex is connective.

We will also consider a more general notion than that of a derived scheme. Namely, that of an n-Artin stack (see §**??**) for a precise definition. The properties that an n-Artin stack $\mathscr{X}$ will have are:

- for any classical scheme S_0 one has $\mathscr{X}(S_0)$ is n-truncated, i.e. $\mathscr{X}|_{\mathsf{CAlg}} : \mathsf{CAlg} \to \text{Spc}$ factors through the subcategory $\text{Spc}^{\leq n}$;
- the cotangent complex $T^*\mathscr{X} \in \text{QCoh}(\mathscr{X})^{\leq n}$.

So, given a prestack $\mathscr{X}$ we can pictorially think of: the connective part $\tau^{\leq 0}(T^*\mathscr{X})$ of the cotangent complex as dealing with the derived structure of $\mathscr{X}$ and the coconnective part $\tau^{\geq 0}(T^*\mathscr{X})$ as regarding the 'stacky' structure of $\mathscr{X}$. We will make this more precise when we discuss the cotangent complex and deformation theory later.

1.2.2 Hidden smoothness

Given a projective curve X and a smooth scheme Y, then one can consider the ordinary scheme $\text{Maps}(X, Y)$ which parametrizes maps $f : X \to Y$. Given a point $f \in \text{Maps}(X, Y)$ one can compute the tangent space at f to be

$$\mathsf{T}_f\text{Maps}(X, Y) \simeq H^0(X, f^*TY).$$

Since $\text{Maps}(X, Y)$ is not smooth, for deformation theory one would like to consider its tangent complex[11] $T\text{Maps}(X, Y)$ which has $H^0(T_f\text{Maps}(X, Y)) \simeq \mathsf{T}_f\text{Maps}(X, Y)$ and $H^1(T_f\text{Maps}(X, Y))$ parametrizes obstructions. In particular, one has

$$T_f\text{Maps}(X, Y) \simeq C^*(X, f^*\mathsf{T}Y). \tag{1.2}$$

Now, suppose we want to consider the situation in which X is a projective surface. The following result shows that there exists no classical scheme $\text{Maps}(X, Y)$ for which a formula (1.2) holds for X a surface.

Theorem 1.2.1 (Avramov)**.** *Let X be a scheme locally almost of finite type, then either one of the following three cases happens:*

- *TX is concentrated in degree 0, i.e. X is smooth;*
- *TX is concentrated in degree $[-1, 0]$, i.e. X is a local complete intersection;*
- *TX is not perfect.*

What the philosophy of hidden smoothness (some names associated to it are: Deligne, Drinfeld, Kontsevich, ...) is that one needs to consider $\text{Maps}(X, Y)$ as a *derived object* and then we will be able to savage the description of (1.2).

[10] We will define the notion of a homotopy category in §2.2.2, for now one should think of $\text{h}\mathscr{C}$ as a 1-categorical shadow of the ∞-category $\mathscr{C}$.

[11] Dual of the cotangent complex.

1.2.3 Non-flat base change

Consider a pullback diagram of derived schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

i.e. $X' \simeq X \underset{Y}{\times}^{\mathrm{L}} Y$. The adjunctions (f^*, f_*) and $((f')^*, f'_*)$ applied to $f_* \circ g'_* \simeq g_* \circ f'_*$ give a base change map

$$f^* \circ g_* \to f^* \circ g_* \circ f'_* \circ (f')^* \simeq f^* \circ f_* \circ g'_* \circ (f')^* \to g'_* \circ (f')^*,$$

which is an isomorphism of functors $\mathrm{QCoh}(Y') \to \mathrm{QCoh}(X)$ (the derived ∞-category of quasi-coherent sheaves), whenever f is *quasi-compact and (quasi-)separated* (see [16, Chapter 3, Proposition 2.2.2])[12]. It is important to notice that there are no flatness assumptions either on f or g. In particular, when X, Y and Y' are classical schemes, one obtains a general base change formula. If either f or g is flat one has

$$X \underset{Y}{\times}^{\mathrm{L}} Y \simeq X \underset{Y}{\times} Y,$$

i.e. the fiber product is actually a classical scheme and we recover the usual flat base change isomorphism for the derived category of quasi-coherent sheaves[13].

1.3 Applications to geometric representation theory

One of the areas that has seen many application of derived geometry is in its connection with representation theory. Usual algebraic geometry, specially its cohomological tools (e.g. intersection cohomology, perverse sheaves, D-modules, etc.) had already shown to be extremely useful in understanding and answering questions in representation theory. Below we list three examples where one is forced to consider not only usual classical algebraic geometry but some derived objects to capture exactly what is happening.

1.3.1 Geometric Langlands Correspondence

Perhaps one of the most famous applications of derived geometry is the enormous project spearheaded by Gaitsgory and many others that seeks to make sense and prove the following statement. Consider G a connected reductive group over $\mathbb{C}$ and X a smooth proper complex curve. One considers two moduli spaces:

$$\mathrm{Bun}_G(X) := \{ \text{ moduli of } G\text{-bundles on } X \}$$

and

$$\mathrm{Loc}_{G^{\mathrm{L}}}(X) := \left\{ \text{ moduli of } G^{\mathrm{L}} \text{ local systems on } X \right\},$$

where G^{L} denotes the Langlands dual group of G[14].

The conjecture states that one has an equivalence of categories

$$D - \mathrm{mod}(\mathrm{Bun}_G(X)) \simeq \mathrm{IndCoh}_{\mathscr{N}}(\mathrm{Loc}_G(X)),$$

where $\mathrm{IndCoh}(\mathrm{Loc}_G(X))$ is a cohomological completion of the category of coherent sheaves on $\mathrm{Loc}_G(X)$ and the subscript $\mathscr{N}$ is a technical condition on the support of sheaves that show up (see [3] for details).

The first comment, is that the categories above need to be considered in the dg sense, i.e. as ∞-categories rather than their homotopy categorical shadow. The second, maybe but more important point, is that the

[12] Notice that without the quasi-compactness assumption the result fails: consider $Y = \mathrm{pt}$, $X = \mathbb{A}^1$ and $Y' = \sqcup_I \mathrm{pt}$, where I is a countable set.

[13] Notice that the functors $f^*, (f')^*, g_*$ and g'_* are understood in the derived sense.

[14] This group can be described by considering the root datum classifying G and taking the group corresponding to the 'dual' root datum. Or (maybe) less mysteriously as the group whose category of representations is the Tannakian category of $G(\mathscr{O})$-equivariant perverse sheaves on the affine Grasmanian of G.

objects $\mathrm{Bun}_G(X)$ and $\mathrm{Loc}_G(X)$ show be treated as stacks and not their coarse moduli spaces. Furthermore, for $\mathrm{Loc}_G(X)$ one needs to actually consider a *derived stack*, i.e. a derived enhancement of the usual moduli space of local systems.

To give a heuristic picture of why local systems need to be derived, let's restrict ourselves to the simplest possible case and take $G = \mathbb{G}_{\mathrm{m}}$. In this case one has

$$\mathrm{Bun}_{\mathbb{G}_{\mathrm{m}}}(X) \simeq \mathscr{P}\mathrm{ic}(X),$$

where $\mathscr{P}\mathrm{ic}(X)$ is the Picard stack of line bundles on X. It is well-know that one has a map

$$\pi : \mathscr{P}\mathrm{ic}(X) \to \mathrm{Pic}(X),$$

where $\mathrm{Pic}(X)$ is the Picard scheme of line bundles on X. Moreover, π is a $B\mathbb{G}_{\mathrm{m}}$-torsor, i.e. locally it is given by a product $\mathrm{Pic}(X) \times \mathrm{B}\,\mathbb{G}_{\mathrm{m}}$. Thus, at first approximation one can try to understand D-modules on the product $\mathrm{Pic}(X) \times \mathrm{B}\,\mathbb{G}_{\mathrm{m}}$ and, yet more concretely, try to understand the category $D-\mathrm{mod}(\mathrm{B}\,\mathbb{G}_{\mathrm{m}})$. The theory of D-modules on stacks, also formalized using the versatile theory of ∞-categories, gives us a descent result that says that:

$$D-\mathrm{mod}(\mathrm{B}\,\mathbb{G}_{\mathrm{m}}) \simeq \lim_{\Delta^{\mathrm{op}}} \left(\mathrm{B}\,\mathbb{G}_{\mathrm{m}}^{\bullet}\right),$$

where $\mathrm{B}\,\mathbb{G}_{\mathrm{m}}^{\bullet}$ is the simplicial object[15] whose colimit in the category of prestacks[16] defines $\mathrm{B}\,\mathbb{G}_{\mathrm{m}}$. Then an argument using the Bar–Beck–Lurie theorem allows one to obtain

$$D-\mathrm{mod}(\mathrm{B}\,\mathbb{G}_{\mathrm{m}}) \simeq \mathrm{Mod}_{\Lambda},$$

where $\Lambda = \mathrm{Sym}(k[1])$. Notice that if one expects this to be recovered from quasi-coherent (or a close enough cousin) on a geometric object this would be

$$D-\mathrm{mod}(\mathrm{B}\,\mathbb{G}_{\mathrm{m}}) \simeq \mathrm{QCoh}(\mathrm{Spec}(\Lambda))$$

where $\mathrm{Spec}(\Lambda)$ is the derived affine scheme corresponding to the commutative differential graded algebra (cdga) $k[\epsilon]/(\epsilon^2)$, where ϵ is in cohomological degree -1.

1.3.2 Mirković–Riche perspective on Koszul duality

Let V be a finite dimensional vector space over a field k, then usual Koszul duality gives an equivalence

$$\mathrm{Mod}(\mathrm{Sym}(V)) = \mathrm{QCoh}(V^*) \simeq \mathrm{QCoh}(\mathrm{pt} \underset{V}{\times}^{\mathrm{L}} \mathrm{pt}) = \mathrm{Mod}(k \underset{\mathrm{Sym}(V^*)}{\otimes^{\mathbb{L}}} k),$$

where $\mathrm{pt} \underset{V}{\times} \mathrm{pt}$ is the derived self-intersection of the origin in the affine space $V \simeq \mathbb{A}^n$.

Mirković and Riche generalized the above to a relative setting. Consider X a smooth Noetherian scheme and E a vector bundle over X. Given $F_1, F_2 \subset E$ two sub-vector bundles of E, they determine orthogonal complements $F_1^{\perp}, F_2^{\perp}$ in E^* the dual vector bundle. Consider a $\mathbb{G}_{\mathrm{m}}$-action on E of weight -2 along the fibers of the canonical projection $E \to X$. Their result then reads:

Theorem 1.3.1. *There exists an equivalence of categories*

$$Coh^{\mathbb{G}_{\mathrm{m}}}(F_1 \underset{E}{\times}^{\mathrm{L}} F_2) \simeq Coh^{\mathbb{G}_{\mathrm{m}}}(F_1^{\perp} \underset{E^*}{\times}^{\mathrm{L}} F_2^{\perp}).$$

The machinery of derived geometry is not so important in proving Theorem 1.3.1, however it frames it in a nice conceptual way. This result was applied by S. Riche to prove an equivalence between different blocks in the category of representations of a Lie algebra $\mathfrak{g}$ of a connected, simply connected, semisimple algebraic group G over an algebraically closed field k of *positive characteristic* (see [45] for details).

[15] One has

$$\mathrm{B}\,\mathbb{G}_{\mathrm{m}}^{n} := \mathbb{G}_{\mathrm{m}}^{\times(n-1)},$$

where the structure maps are a combination of the canonical projections and multiplication maps.

[16] To obtain a stack one needs to sheafify the resulting prestacks, however that doesn't change the category of D-modules, so this description is good enough for our discussion.

1.3.3 Geometric Affine Hecke algebra

Given a reductive group G, an object of interest in representation theory in the past decade or so is the affine Hecke algebra $\mathscr{H}_G$ associated to G. Actually, what is more relevant from a modern point of view is to consider a categorification of $\mathscr{H}_G$, i.e. a certain category whose Gronthedieck K-group recovers it. One natural candidate is to consider D-modules on the following object[17]

$$I\backslash G(\mathscr{K})/I$$

where $G(\mathscr{K})$ is the loop group associated to G and $I \subset G(\mathscr{K})$ is the Iwahori subgroup.

Based on work of Kazhdan–Lusztig and Ginzburg, Bezrukavnikov [9] also considered a category of coherent sheaves on the scheme

$$\widetilde{\mathscr{N}} \underset{\mathfrak{g}^{\mathrm{L}}}{\times} \widetilde{\mathscr{N}},$$

where $\mathfrak{g}^{\mathrm{L}} = \mathrm{Lie}(G^{\mathrm{L}})$ is the Lie algebra of the Langlands dual group and $\widetilde{\mathscr{N}}$ is the Springer resolution of the nilpotent cone of $\mathfrak{g}^{\mathrm{L}}$[18]

Bezrukavnikov then proves that

$$D - \mathrm{mod}(I\backslash G(\mathscr{K})/I) \simeq \mathrm{Coh}^{G^{\mathrm{L}}}(\widetilde{\mathscr{N}} \underset{\mathfrak{g}^{\mathrm{L}}}{\times} \widetilde{\mathscr{N}}), \tag{1.3}$$

where one considers G^{L}-equivariant coherent sheaves on the right-hand side.

Again the important qualifier here to make the equivalence (1.3) correct, besides defining the category on the left-hand side properly, is that we should consider not the classical scheme $\widetilde{\mathscr{N}} \underset{\mathfrak{g}^{\mathrm{L}}}{\times} \widetilde{\mathscr{N}}$ but its natural derived enhacement obtained by taking the *derived* fiber product.

1.4 Applications to Enumerative geometry

1.4.1 Gromov–Witten invariants

In the study of Gromov–Witten invariants and other questions in enumerative geometry often one considers the moduli space $\mathsf{M}_g(S)$ of stable maps from a curve of genus g into a smooth proper scheme X. Many people (for instance [5]) developed and studied the notion of a (perfect) obstruction theory on $\mathsf{M}_g(X)$. One conceptual way of understanding the data of an obstruction theory is to find a derived stack $\mathscr{M}_g(X)$ whose underlying classical stack is equivalent to $\mathsf{M}_g(X)$, i.e. ${}^{c\ell}\mathscr{M}_g(X) \simeq \mathsf{M}_g(X)$. Thus, by adjunction we have a canonical map[19]

$$\jmath : \mathrm{LKE}_{c\ell}(\mathsf{M}_g(X)) \to \mathscr{M}_g(X).$$

The following is a modern formulation of an obstruction theory ([5, Definition 4.4]):

Definition 1.4.1. A (resp. *perfect*) *obstruction theory* for a classical prestack $\mathscr{X}_0$ is the data of a (resp. compact) of Tor amplitude $[-1, 0]$ (see the next subsection for what this means) object $\mathscr{E} \in \mathrm{QCoh}(\mathscr{X}_0)$ and a morphism

$$\varphi : \mathscr{E} \to T^*\mathscr{X}_0$$

such that $\mathrm{Cofib}\varphi \in \mathrm{QCoh}(\mathscr{X}_0)^{\leq -2}$, i.e. $\mathrm{Cofib}\,\varphi$ is 2-connective.

In [49, Proposition 1.2] the authors show that

$$\jmath^* T^* \mathscr{M}_g(X) \to \mathrm{LKE}_{c\ell}(\mathsf{M}_g(X)) \tag{1.4}$$

is an obstruction theory. Moreover tautologically, if $\mathscr{M}_g(X)$ is quasi-smooth, i.e. $T^*\mathscr{M}_g(X)$ is perfect of Tor-amplitude $[-1, 0]$, then the obstruction theory from (1.4) is perfect.

[17] We don't want to get into the details of what kind of stack, or scheme this is at this moment.

[18] I.e. $\mathscr{N} = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in G^{\mathrm{L}}/B,\ x \in \mathrm{rad}(\mathfrak{b})\}$, where $\mathrm{rad}(\mathfrak{b})$ denotes the radical of $\mathfrak{b}$.

[19] Here on the right-hand side $\mathrm{LKE}_{c\ell}$ is the functor that takes a classical scheme and regards it as a derived scheme. We give more details on this in §**??**.

1.4.2 Virtual fundamental class

Another important ingredient in enumerative geometry is the virtual fundamental class of certain moduli spaces (we refer the reader to [25] and [54, §3.1] for more details). Before we explain the heuristics, we introduce a construction which is somewhat analogous to that of the relative spectrum in usual algebraic geometry.

Let $\mathscr{X}$ be a derived Artin stack and $\mathscr{E} \in \mathrm{Perf}(\mathscr{X})$ a perfect complex with *Tor-amplitude*[20] *contained in* $[-1, k]$, i.e. for any $\mathscr{F} \in \mathrm{QCoh}(\mathscr{X})^{\heartsuit}$ one has

$$\mathscr{E} \otimes \mathscr{F} \in \mathrm{QCoh}(\mathscr{X})^{[-1,k]}.$$

Given $\mathscr{X}$ any derived Artin stack and $\mathscr{E} \in \mathrm{QCoh}(\mathscr{X})$, one defines $\mathbb{V}(\mathscr{E})$ the *linear stack associated to* $\mathscr{E}$ as follows: for any map $u : S \to \mathscr{X}$ from a derived affine scheme S let

$$\mathbb{V}(\mathscr{E})(S) := \mathrm{Hom}_{\mathrm{QCoh}(S)}(u^*\mathscr{E}, \mathscr{O}_S).$$

Notice that $\pi : \mathbb{V}(\mathscr{E}) \to \mathscr{X}$ is a stack over $\mathscr{X}$ by simply just remembering the data of u and it also has a zero section $s_0 : \mathscr{X} \to \mathbb{V}(\mathscr{E})$ given by considering the trivial morphisms between these coherent sheaves. Here are a couple of properties of this construction:

Proposition 1.4.2. *(a) If $\mathscr{E}$ is a compact object, i.e. $\mathscr{E} \in Perf(\mathscr{X})$, $\mathbb{V}(\mathscr{E})$ is an Artin stack;*

(b) One has an equivalence $\pi^*(T(\mathbb{V}(\mathscr{E})/\mathscr{X})) \simeq \mathscr{E}^\vee$;

(c) If $\mathscr{E}$ has Tor-amplitude in $[0, k]$[21] then $\mathbb{V}(\mathscr{E}) \to \mathscr{X}$ is k-representable, i.e. its fibers are k-Artin stacks;

(d) $\pi : \mathbb{V}(\mathscr{E}) \to \mathscr{X}$ is smooth if and only if $\mathscr{E}$ has Tor-amplitude in $[-\infty, 0]$.

In particular, let $f : \mathscr{X} \to \mathscr{Y}$ be a k-representable quasi-smooth morphism of derived stacks, i.e. $T^*(\mathscr{X}/\mathscr{Y}) \in \mathrm{Perf}(\mathscr{X})$ and has Tor-amplitude $[-1, k]$. Then

$$\mathbb{V}_{\mathscr{X}}(T^*(\mathscr{X}/\mathscr{Y})[-1]) \to \mathscr{X}$$

is a smooth relative $(k+1)$-stack.

In particular, if f is a closed immersion, then $T^*(\mathscr{X}/\mathscr{Y})[-1]$ has Tor-amplitude in $[0, 0]$ and $\mathbb{V}_{\mathscr{X}}(T^*(\mathscr{X}/\mathscr{Y})[-1])$ is the normal bundle. In the case that f is smooth $T^*(\mathscr{X}/\mathscr{Y})[-1]$ has Tor-amplitude in $[-\infty, -1]$ and $\mathbb{V}_{\mathscr{X}}(T^*(\mathscr{X}/\mathscr{Y})[-1])$ is the classifying stack of the tangent bundle. If one factors f as a closed immersion $\imath : \mathscr{X} \to \mathscr{Y}'$ followed by a smooth morphism $f : \mathscr{Y}'\mathscr{Y}$ one has

$$\mathbb{V}_{\mathscr{X}}(T^*(\mathscr{X}/\mathscr{Y})[-1]) \simeq [\mathbb{V}_{\mathscr{X}}(T^*(\mathscr{X}/\mathscr{Y}')[-1])/\imath^*T(\mathscr{Y}'/\mathscr{Y})].$$

Let's get back to the situation of

$$\jmath : \mathsf{M}_g(X) \to \mathscr{M}_g(X),$$

where we write $\mathsf{M}_g(X)$ for $\mathrm{LKE}_{c\ell}\mathsf{M}_g(X)$, to consider the classical moduli space $\mathsf{M}_g(X)$ as a derived stack. Notice that one has a canonical morphism:

$$\mathbb{V}(T^*\mathsf{M}_g(X)) \to \mathbb{V}(\jmath^*T^*\mathscr{M}_g(X)).$$

The *fundamental class of* $\mathsf{M}_g(X)$ is defined as the class (in an appropriate cohomology theory[22]) of the derived intersection

$$\mathsf{M}_g(X) \underset{\mathbb{V}(\jmath^*T^*\mathscr{M}_g(X))}{\times} \mathbb{V}(T^*\mathsf{M}_g(X)).$$

[20] Here we are using the cohomological indexing convention.

[21] We remind the reader that (with cohomological convention) a element $\mathscr{E} \in \mathrm{QCoh}(\mathscr{X})$ has *Tor-amplitude contained in* $[a, b]$ if for any $\mathscr{F} \in \mathrm{QCoh}(\mathscr{X})^{\heartsuit}$ one has

$$\mathscr{E} \otimes \mathscr{F} \in \mathrm{QCoh}(\mathscr{X})^{[a,b]}.$$

[22] Picking the correct one and formalizing it in this level of generality is one of the problem, see [25] where this is done using motivic homotopy theory.

1.5 Homotopy theory

The theory of derived geometry, including also derived stacks and not only derived schemes, helps us to better understand certain aspects of homotopy theory or the cohomology of schemes.

1.5.1 Étale cohomology

Given a scheme X an *Azumaya algebra over* X is a locally free sheaf $\mathscr{A}$ of finite rank such that

$$\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}} \to \mathscr{E}\mathrm{en}_X(\mathscr{A})$$

is an isomorphism.

Given an Azumaya algebra $\mathscr{A}$ one can construct an étale cohomology class $[\mathscr{A}] \in H^2(X, \mathbb{G}_{\mathrm{m}})$ the so-called Brauer group of X. Gabber ([14]) proved that when X is qcqs any *torsion* class in $H^2(X, \mathbb{G}_{\mathrm{m}})$ can be represented by an Azumaya algebra.

Toën ([52]) generalized this result to all (not necessarily torsion) classes in $H^2(X, \mathbb{G}_{\mathrm{m}})$ by considering *derived Azumaya algebras*, i.e. $\mathscr{A}$ is now a perfect complex in $\mathrm{QCoh}(X)$.

1.5.2 Rational homotopy theory

Our exposition in this section is entirely motivated by [28], we refer the reader to the original for an exposition of rational homotopy theory from derived geometry. Let $\mathrm{Spc}_*^{\mathrm{rat}}$ denote the subcategory of the ∞-category of spaces X which satisfy the following:

- X is simply connected;
- for every $n \geq 2$ the abelian group $\pi_n(X)$ (which doesn't depend on a choice of a point in X) is a rational vector space.

These are called *rational* topological spaces. Work of Quillen ([42]) allowed one to understand rational topological spaces from an algebraic point of view:

Theorem 1.5.1. *The category Spc_*^{rat} is equivalent to the ∞-category $Lie_{\mathbb{Q}}^{\leq -1}$ of connected[23] differential graded Lie algebras over the rational numbers.*

After applying Koszul duality (or the result on representability of formal moduli problems) the category $\mathrm{Lie}_{\mathbb{Q}}^{\leq -1}$ is actually related to a certain ∞-category of commutative differential graded algebras.

For X a topological space, let $C^*(X;\mathbb{Q})$ denote the commutative algebra object[24] in $\mathrm{Vect}_{\mathbb{Q}}$ given by singular cochains on X. The construction $X \mapsto C^*(X;\mathbb{Q})$ admits a right adjoint which gives the following adjunction

$$C^*(-;,\mathbb{Q}) : \mathrm{Spc} \rightleftarrows \mathrm{CAlg}(\mathrm{Vect})^{\mathrm{op}} : \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Vect})}(-,k). \tag{1.5}$$

For X a rational topological space, Sullivan ([51]) gave a model of a commutative differential graded algebra $C^*(X;\mathbb{Q})$ which encodes the data of the differential graded Lie algebra associated to the space X. His result can be roughly stated as:

Proposition 1.5.2. *For any object $X \in Spc_*^{\mathrm{rat}}$ the canonical unit map from the adjunction (1.5)*

$$X \to Hom_{CAlg}(C^*(X;\mathbb{Q}),\mathbb{Q})$$

is an isomorphism in Spc, i.e. it is a homotopy equivalence. In particular, the restriction of $C^*(-;\mathbb{Q})$ to $Spc_{\mathbb{Q}}^{\mathrm{rat}}$ gives a fully faithful functor

$$Spc_{\mathbb{Q}}^{\mathrm{rat}} \hookrightarrow CAlg(Vect_{\mathbb{Q}}).$$

[23] By definition connected Lie algebras have vanishing cohomology in degrees $kgeq0$.

[24] This is a homotopically coherent multiplication. One way to formalize this is we consider the functor of ∞-categories:

$$X \overset{k_X}{\to} \mathrm{Vect}_k,$$

which sends all points of X to $k \in \mathrm{Vect}_k$. The vector space $C^*(X;k)$ is identified with the functor $C^*(X;k) : \mathrm{pt} \to \mathrm{Vect}$ obtained by right Kan extension of k_X via the canonical projection $X \to \mathrm{pt}$. It is easy to see that this construction is lax symmetric monoidal, hence $C^*(X;k)$ has the structure of a commutative algebra.

Another perspective on Proposition 1.5.2 is that one is trying to understand which prestacks $\mathscr{X}$: $\mathrm{CAlg}(\mathrm{Vect}_{\mathbb{Q}}^{\leq 0}) \to \mathrm{Spc}$ are co-representable in $\mathrm{CAlg}(\mathrm{Vect}_{\mathbb{Q}})$, i.e. there exists $A \in \mathrm{CAlg}$ such that

$$\mathscr{X}(R) \simeq \mathrm{Hom}_{\mathrm{CAlg}}(A, R)$$

for all $R \in \mathrm{CAlg}(\mathrm{Vect}_{\mathbb{Q}}^{0})$. These are called *coaffine stacks*. Theorem 1.5.1 and Proposition 1.5.2 can be rephrased[25] to say that any prestack that is co-representable by an algebra $A \in \mathrm{CAlg}(\mathrm{Vect}^{\geq 0})$ such that $H^1(A)$ vanishes and $H^i(A)$ are finite-dimensional, is equivalent to $\mathrm{Hom}_{\mathrm{CAlg}}(C^*(X;\mathbb{Q}), -)$ for some rational topological space X.

This whole discussion can be performed for $\mathrm{CAlg}(\mathrm{Vect}_k)$ where k is an algebraically closed field of characteristic p this was first developed by Mandell in [36].

1.5.3 Elliptic cohomology

Abstract homotopy theory studies the interesting concept of a *cohomology theory*. Roughly that means a collection of functors $\{A^n : \mathrm{Spc}^{\times 2} \to \mathrm{Ab}\}_{n\in\mathbb{Z}}$ from the category of pairs of topological spaces to the category of abelian groups, plus the data of connecting morphisms ∂^n. This data is required to satisfy axioms roughly encoding: (i) long exact sequences associated to a triple of spaces $Z \subset Y \subset X$; (ii) excision, i.e. $A^n(X, Y) \simeq A(X \backslash U, Y \backslash U)$ for an open $U \subseteq X$ s.t. $\overline{U} \subseteq Y$; (iii) $A^n(-, \emptyset)$ send coproducts to products and (iv) normalization, i.e. $A^n(\mathrm{pt}) \simeq \mathbb{Z}$ if $n = 0$.

Surprisingly for a class of cohomology theories a lot of the data of A^* can be recovered from the object

$$A^*(\mathbb{CP}^\infty) \simeq A^*(\mathrm{pt})[[t]]$$

where the above isomorphism is non-canonical. To actually recover all of A^* from $A(\mathrm{pt})[[t]]$ one needs to consider the extra structure of a group structure on the formal scheme $\mathrm{Spf}(A(\mathrm{pt})[[t]])$, i.e. the data of a so-called formal group law.

Since homotopy theorists want to understand cohomology theories and the discussion above says that one can understand certain cohomology theories from formal group laws; they are naturally interested in studying formal group laws. Formall group laws can be rather complicate to understand–they are stratified by dimension and height, thus we could focus on the one-dimensional case for a start. It turns out that all one-dimensional formal group laws arrive from completions of one-dimensional algebraic groups at the identify. There are essentially three examples of one-dimensional algebraic groups: $\mathbb{G}_{\mathrm{a}}$, $\mathbb{G}_{\mathrm{m}}$ and E, where E is an elliptic curve. The last example actually provides many different possibilities over $\mathbb{Z}$. A cohomology theory A associated to an elliptic curve E is essentially characterized by the following list of axioms:

a) the data of a commutative ring R;

b) E an elliptic curve over R;

c) a multiplicative cohomology theory A satisfying some technical conditions;

d) isomorphisms
$$A(\mathrm{pt}) \simeq R \quad \text{and} \quad \hat{E} \simeq \mathrm{Spf}(A^0(\mathbb{CP}^\infty))$$
where $\hat{E}$ denotes the formal completion of E at the identity.

Derived algebraic geometry packages all the above data into the following picture. Let $\mathscr{M}_{1,1}$ denote the moduli of elliptic curves. There is an unique (étale) sheaf $\mathscr{O}^+$ of $\mathbb{E}_\infty$-rings on $\mathscr{M}_{1,1}$ with the property that: given an étale map $p : \mathrm{Spec}(R) \to \mathscr{M}_{1,1}$ where R is a commutative ring, let E_p denote the corresponding elliptic curve and $A_p := \mathscr{O}^+(\mathrm{Spec}(R))$, then one has

$$H^0(A) \simeq R \quad \text{and} \quad \mathrm{Spf}(A^0(\mathbb{CP}^\infty)) \simeq \hat{E}.$$

The moduli space $\mathscr{M}_{1,1}$ and its sheaf of derived rings $\mathscr{O}^+$ can be used to formulate what the theory of topological modular forms are, essentially they are given by the global sections of $\mathscr{O}^+$.

[25] This is one way to formalize the ∞-category of rational homotopy types.

The construction of elliptic cohomology has been an active area of research for years. A definitely non-exhaustive list of references for the reader interested in more is [8, 29, 30, 31, 19]. It is interesting to notice that to formulate an equivariant version of the theory the theory of ∞-categories is also very useful to make sense of the source category, namely what is the correct notion of spaces with a G-action.

1.6 Symplectic geometry

1.6.1 More robust theory

Normally, symplectic geometric on starts by considering a smooth scheme X with an isomorphism

$$\omega_X^\natural : TX \xrightarrow{\simeq} T^*X.$$

However, this set up can be restrictive when one is considering non-smooth schemes or stacks. Shifted symplectic geometry starts with the simple observations that one should consider geometric objects $\mathscr{X}$ with an isomorphism

$$T\mathscr{X} \simeq T^*\mathscr{X}[n],$$

where $T\mathscr{X}$ and $T^*\mathscr{X}$ are the tangent and cotangent complexes and $n \in \mathbb{Z}$ is the shift necessary to pair these two objects. It turns out that one can make sense of this theory for very general objects, $\mathscr{X}$ a derived Artin stack locally almost of finite type. Moreover, many constructions of usual symplectic geometry: symplectic and Hamiltonian reduction, Lagrangian intersections, critical locus of a 1-form, and etc; can be generalized to this setting and also clarified by it. See [40] for the foundations of this theory and [54, §5] for a summary.

1.6.2 Symplectic reduction

Let X be a smooth scheme with a Hamiltonian action of a group G. One can interpret the usual symplectic reduction as a derived intersection

$$X//G \simeq [X/G] \underset{\mathfrak{g}^\vee/G}{\times} BG$$

where $BG \to \mathfrak{g}^\vee/G$ is the inclusion of the origin of $\mathfrak{g}^\vee$. However, there are more general construction where one considers any element $\lambda \in \mathfrak{g}^\vee$, in particular, one doesn't need to assume that λ is a regular value of the moment map $\phi : T^*X \to \mathfrak{g}^\vee$. See [47, 48] for more details.

1.6.3 Batalin–Vilkovisky formalism

Let M be a smooth algebraic variety, sometimes thought of as the space of field of fields of physical theory. Given a function $f \in \mathscr{O}(M)$ in physics known as an action functional. One is often interested in understanding the solutions to $df = 0$. This can be realized as the following derived fiber product

$$\begin{array}{ccc} \mathrm{Crit}(f) & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle df} \\ M & \xrightarrow{0} & T^*M \end{array}$$

Concretely, one can describe $\mathrm{Crit}(f)$ as the derived scheme whose space is given by $T^*[-1]M := \mathbb{V}(TM[1])$[26] and with structure sheaf the complex of sheaves given by $\mathrm{Sym}^\bullet(TM[1])$ together with a differential given by $\wedge df$. This is the base of the so-called Batalin–Vilkovisky formalism in mathematical physics that has been very successful in describing a big number of quantum field theories and their classical limits. Moreover, the derived critical locus carries a (-1)-*shited sympletic structure*, roughly speaking this means one has a (symmetric) isomorphism

$$T\mathrm{Crit}(f) \simeq T^*\mathrm{Crit}(f)[-1].$$

A case of particular relevance is when $M = Y/\mathfrak{g}$, where Y is an affine space and $\mathfrak{g}$ a Lie algebra. In this case this formalism has been worked out in [12, Chapter 5] and [58].

[26] Recall this is informally $\mathrm{Spec}_M(\mathrm{Sym}^\bullet(TM[1]))$.

Chapter 2

∞-categories

2.1 Introduction

Before discussing some derived algebraic geometry we will need a crash course in the theory of $(\infty, 1)$-categories.

2.1.1 Why ∞-categories?

There are a couple of reasons for that:

I. the main geometric objects that one is interested in DAG will be organized into ∞-categories for example: derived (affine) schemes, derived stacks, prestacks, etc. Here area some reasons why one needs to have the right categorical description of these objects: (i) correct statement of universal properties; (ii) appropriate notion of Grothendieck topology and locally $\mathbb{E}_\infty$-ringed topoi; (iii) sometimes one uses a categorical argument to obtain an algebro-geometric result, for instance Neeman's construction of Grothendieck duality (see [39]).

II. the very definition of certain objects in derived geometry needs one to deal with ∞-categories properly: what is the correct notion of an $\mathbb{E}_\infty$-object?

III. very often we will be interested in studying sheaves over a geometric object, or more generally complexes of sheaves up to quasi-isomorphism, i.e. some version of a derived category. However, to obtain many desirable properties for these sheaves one needs to consider a more refined version of the derived category. Indeed, here is a concrete example that shows that triangulated categories are not enough: consider $\mathbb{P}^1$ and let $D(\mathbb{P}^1)$ denote its derived category of quasi-coherent sheaves. Then it is *not* the case that we can glue the derived category from its value on an open cover, i.e. $D(\mathbb{P}^1) \not\simeq \lim_{\Delta^{\mathrm{op}}}(D(U/X)^\bullet)$, where $U \hookrightarrow X$ is a Zariski cover of X. Notice that
$$\mathbb{R}\mathrm{Hom}_{D(\mathbb{P}^1)}(\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1}(-2)[1]) \simeq H^0\mathbb{R}\Gamma(\mathscr{O}_{\mathbb{P}^1}(-2)[1]) \simeq H^1(\mathscr{O}_{\mathscr{P}^1}(-2)) \simeq k,$$
where the last isomorphism follows from Serre duality:[1] $H^1(\mathscr{O}_{\mathscr{P}^1}(-2)) \simeq H^0(\omega_{\mathbb{P}^1} \otimes \mathscr{O}_{\mathscr{P}^1}(-2)^\vee)^\vee$. It will however be true that
$$\mathrm{QCoh}(\mathbb{P}^1) \simeq \mathrm{QCoh}(U_0) \underset{\mathrm{QCoh}(U_{01})}{\otimes} \mathrm{QCoh}(U_1),$$
when we interpret the fiber product of the above ∞-categories in the appropriate sense of a fiber product of ∞-categories.

IV. In the discussion of §1.1.2 the notion of space-valued sheaf that we want to consider needs to be up to homotopy, thus one needs to treat the collection of topological spaces as an ∞-category Spc.

[1]Recall that $\omega_{\mathbb{P}^1} \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$ (see [57, §30.1.6]).

2.1.2 A brief discussion of models

By an ∞-category (short for $(\infty, 1)$-category) we will mean a category that has n-morphisms for all $n \geq 1$ which are invertible for all $n \geq 2$. There are many concrete ways to define a theory[2] of such. There are many options for a definition of an ∞-category, for instance:

(i) *Segal categories*: a simplicial space $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Spc})$ satisfying:

- (discrete 0th space): $X_0 \in \mathrm{Spc}_{\leq 0}$, i.e. X_0 is a discrete topological space;
- (Segal condition): for every $n \geq 2$ one has an isomorphism[3]

$$X_n \xrightarrow{\simeq} X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1.$$

morphisms are maps of simplicial spaces[4].

(ii) *topological categories*: categories enriched in topological spaces[5];

(iii) *simplicial categories*: categories enriched over simplicial sets, morphisms $F : \mathscr{C} \to \mathscr{D}$ are such that $\mathrm{Maps}_{\mathscr{C}}(X, Y) \to \mathrm{Maps}_{\mathscr{D}}(F(X), F(Y))$ is a map of simplicial sets[6];

(iv) *quasi-categories*[7]: is a simplicial set K such that every diagram as follows

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

has a lift, i.e. the dotted arrow exists, for any $n \geq 0$ and $1 < i < n$; morphisms of quasicategories are maps of simplicial sets[8];

[2]By a theory we mean two things:

- a notion of an ∞-category;
- a notion of the $(\infty, 2)$-category (or at least ∞-category) of (small) ∞-categories.

[3]This condition is equivalent to the Segal condition spelled out in the definition of complete Segal spaces.

[4]This model is a bit trickier, since there are two relevant model structures in the category of Segal precategories, i.e. simplicial spaces that only satisfy the discrete 0th space condition, (see [7, §6.3 and 6.5]): they are roughly characterized as follows:

– weak equivalences: Dwyer–Kan equivalences (see [7, §5.2] for details);
– fibrant objects: (i) (Reedy fibrant) Segal categories, or (ii) Segal categories fibrant in the projective model structure on simplicial spaces;
– cofibrations are either: (i) monomorphisms, or (ii) a smaller collection obtained as certain pushouts (see [7, Theorem 6.5.1]).

[5]The model structure is given by

- weak equivalences: Dwyer–Kan equivalences (defined as for simplicial categories);
- fibrations: $F : \mathscr{C} \to \mathscr{D}$ such that: (i) $\mathrm{Maps}_{\mathscr{C}}(X, Y) \to \mathrm{Maps}_{\mathscr{D}}(F(X), F(Y))$ is a (Serre) fibration and (ii) for every weak equivalence $e : F(X) \to Y'$ in $\mathscr{D}$ (i.e. induces an isomorphism on $\pi_0\mathscr{D}$), there exists a weak equivalence $d : X \to Y$ in $\mathscr{C}$ such that $F(d) = e$;
- cofibrations: harder to describe see [22, Theorem 2.4].

[6]The model structure is given by:

- weak equivalences: Dwyer–Kan equivalences, i.e. a simplicial functor $F : \mathscr{C} \to \mathscr{D}$ such that: (i) for every objects $X, Y \in \mathscr{C}$ the map $\mathrm{Maps}_{\mathscr{C}}(X, Y) \to \mathrm{Maps}_{\mathscr{D}}(F(X), F(Y))$ is a weak equivalence (induced map of topological spaces is a weak homotopy equivalence) of simplicial sets; (ii) the induced functor $\pi_0\mathscr{C} \to \pi_0\mathscr{D}$ ($\pi_0\mathscr{C}$ is the ordinary category obtained by taking the connected component of the mapping simplicial sets) is an equivalence of categories;
- fibrations: defined as for topological categories;
- cofibrations: see [7, Theorem 4.3.2].

[7]These were initially introduced by Boardman and Vogt in [11], they are also refereed to as: *weak Kan complexes* or *inner Kan complex.*

[8]There is model structure on the category of simplicial sets described as:

(v) *complete Segal spaces*[9]: a simplicial space $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Spc})$ satisfying:

- (Segal condition): for every $n = n_1 + n_2$ one has an isomorphism

$$X_n \xrightarrow{\simeq} X_{n_1} \underset{X_0}{\times} X_{n_2},$$

where the maps $X_{n_1} \to X_0 \leftarrow X_{n_2}$ are given by $0 \in [0] \mapsto n_1 \in [n_1]$ and $0 \in [0] \mapsto 0 \in [n_2]$;

- (complete condition): the subspace $X_1^{\mathrm{inv}} \subseteq X_1$ of invertible[10] morphisms is weakly equivalent to X_0;

; morphisms of complete Segal spaces are simply maps of simplicial spaces[11].

(vi) *relative categories*: this is a pair $(\mathscr{C}, \mathrm{W})$ where $\mathscr{C}$ is a category and W is a collection of morphism in $\mathscr{C}$ called weak equivalences.

Some of the models above are better, namely

2.1.3 Higher category theory

Once one makes a choice from above, to defined the correct object that encodes all ∞-categories with a certain size restriction one needs to endow the ordinary category of the above objects with a model structure. s obtained is sometimes refereed to as a homotopy theory of ∞-categories. Many of those turn out to be equivalent (see [7] for a nice discussion of the equivalence between models 1 and 3-6 [12]).

We will whenever possible avoid picking one of the models above. For concreteness the reader is encourage to consider quasi-categories as the running model, for the convenience that the reference [33] provides when one needs many categorical results.

After picking one of these theories one would like to use it just as ordinary category theory, i.e. perform constructs as: given two objects consider the Hom-space between them, compose morphisms, take adjoints, check universal properties, take left or right Kan extensions, ...

We will approach this theory as follows. In §2.2 we give concrete definitions for the basic objects of the theory in the model of quasi-categories. In §2.3 we quickly summarize the analogues of classical categorical concepts in ∞-categories that we will need later in these notes. We will try to formulate these concepts in as a much a model-independent way as possible, so that the reader that has a different model than quasi-categories in mind can follow adapt the discussion.

- weak equivalences: are *Joyal equivalences*, i.e. $f : A \to B$ a map of simplicial sets such that

$$\mathrm{h}A \to \mathrm{h}B$$

is an equivalence of $\mathscr{H}$-enriched categories, here $\mathrm{h}A$ denotes the homotopy category associated to the simplicial category $\mathfrak{C}[A]$;

- fibrant objects: are quasi-categories;
- cofibrations: monomorphisms, i.e. degreewise injections.

[9]See [7, §3.3] for a discussion that illuminates this definition.

[10]A morphism $\alpha \in X_1$ is *invertible* if there exists $\beta \in X_1 \underset{X_0 \times X_0}{\times} (t(\alpha), s(\alpha))$ ($t, s : X_1 \to X_0$ are the target and sources structure morphisms) such that $\alpha \circ \beta$ and $\beta \circ \alpha$ (the composition is defined using the first condition, i.e. $X_2 \simeq X_1 \underset{X_0}{\times} X_1$) belong to the essential image of the degeneracy map $X_0 \to X_1$.

[11]There is a model structure in the category of simplicial spaces (see [7, Theorem 5.3.3]) characterized as:

- weak equivalences: $f : X \to Y$ such that for any Z a complete Segal space the morphism

$$\mathrm{Maps}(Y, Z) \to \mathrm{Maps}(X, Z)$$

is a weak equivalence of spaces;

- fibrant objects: complete Segal spaces;
- cofibrations: monomorphisms, i.e. levelwise injections.

[12]See [33] for a more extensive discussion of model 2 together with the proof that it is compatible with model 4.

2.2 Basic definitions

In this section we present concrete definitions of the notions of ∞-categories, the mapping space between objects, functors between ∞-categories, constructions to produce interesting and important ∞-categories from more strict data, e.g. categories enriched in simplicial sets, and finally we give definitions of the ∞-categories of spaces and ∞-categories themselves. All of this is done in the model of quasi-categories for concreteness. Our main references for this material are [33, 34, 44].

2.2.1 Quasi-categories

Definition 2.2.1 ([34, Tag 003A])**.** An *∞-category* is a simplicial set $S_\bullet : \Delta^{\mathrm{op}} \to \mathbf{Set}$ satisfying the property that for all $n \geq 2$ and for all $0 < i < n$ the following dotted arrow exists

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S_\bullet \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array} \tag{2.1}$$

where $\Lambda^n_i \subset \Delta^n$ denotes the ith horn[13].

Example 2.2.2. (a) Kan complexes, i.e. a simplicial set $S_\bullet$ satisfying (2.1) *but* for all $0 \leq i \leq n$ and $n \geq 0$. In particular, given a topological space X the simplicial set $\mathrm{Sing}_\bullet(X)$[14] is a Kan complex.

(b) given any ordinary category C its *nerve* $\mathrm{N}_\bullet\mathsf{C}$[15] is an example of a simplicial set.

(c) products and coproducts of ∞-categories are ∞-categories (see [34, Tag 0039] for details).

Exercise 2.2.3. Prove that a simplicial set $S_\bullet$ satisfy (2.1) with an unique dotted arrow filling the diagram (2.1) if and only if it is the nerve of an ordinary category.

Remark 2.2.4. Given an ∞-category $\mathscr{C}$ by an *object X of $\mathscr{C}$* we will mean a map $x : \Delta^0 \to \mathscr{C}$, i.e. X[16] is a vertex of the corresponding simplicial set. We will simply write $X \in \mathscr{C}$ to mean that X is an object of $\mathscr{C}$.

Similarly, a *morphism $f : X \to Y$ in $\mathscr{C}$* will mean a map $f : \Delta^1 \to \mathscr{X}$, i.e. f is an edge of the corresponding simplicial set.

The following notion is important for concretely understanding how to pass from an ∞-category to an ordinary category.

Definition 2.2.5. Given two morphisms $f, g : X \to Y$ in $\mathscr{C}$ a *homotopy between f and g* is the data of a map $\sigma : \Delta^2 \to \mathscr{C}$ whose image in $\mathscr{C}$ is the following 2-simplex

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{\mathrm{id}_Y}{\searrow} & \\ X & & \xrightarrow{\quad g \quad} & & Y. \end{array}$$

Exercise 2.2.6. (i) Let $\hom_{\mathscr{C}}(X, Y)$ denote the set of morphisms between the objects X and Y. Prove that the existence of homotopy between morphisms is an equivalence relation on $\hom_{\mathscr{C}}(X, Y)$. We will denote this equivalence relation by $f \sim g$.

[13] I.e. the simplicial subset of Δ^n obtained by discarding the $(n-1)$-face opposite to the ith vertex and the interior of Δ^n.

[14] For every $n \geq 0$

$$\mathrm{Sing}_n(X) := \{\text{continuous functions } f : |\Delta^n| \to X\},$$

where $|\Delta^n|$ is the topological n-simplex (See [34, Tag 001Q] for more details).

[15] Recall that this is defined as

$$N_n(\mathsf{C}) := \{ \text{ functors } F : [n] \to \mathsf{C}\},$$

where $[n]$ is the linearly ordered set $\{0 < 1 < \cdots < n\}$.

[16] For any $n \geq 0$ we let Δ^n denote the standard n-simplex, i.e. the simplicial set defined by

$$\Delta^n_m := \mathrm{Hom}_\Delta([m], [n]).$$

(ii) For C a 1-category, show that given two morphisms f, g in $N_\bullet \mathsf{C}$ then $f \sim g$ if and only if $f = g$.

(iii) Show that $f \sim g$ if and only if there exists a map $\Delta^1 \times \Delta^1 \to \mathscr{C}$ corresponding to the following 2-simplex

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & \searrow^{h} & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow{g} & Y. \end{array}$$

Remark 2.2.7. One of the main difficulties in the theory of ∞-categories is that the composition of morphisms is not uniquely defined. However, the equivalence class of the composite is well-defined. Indeed, given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$ we will say that h is *a composition of f and g* if there exists a map of simplicial sets $\sigma : \Delta^2 \to \mathscr{C}$ whose image is

$$\begin{array}{ccccc} & & Y & & \\ & {}^{f}\nearrow & & \searrow^{g} & \\ X & & \xrightarrow{\quad h \quad} & & Z. \end{array} \tag{2.2}$$

Exercise 2.2.8. Prove that in the remark above, given two σ, σ' whose restriction to the $\{0, 2\}$ edge is $h : X \to Z$ and $h' : X \to Z$, then $h \sim h'$, i.e. there exists a homotopy between h and h'.

Definition 2.2.9. Given $\mathscr{C}$ an ∞-category define its *homotopy category* $\mathrm{h}\mathscr{C}$ to be the category whose

- objects are the same as the objects of $\mathscr{C}$;
- given $X, Y \in \mathrm{h}\mathscr{C}$ the morphism set $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X, Y)$ is the set of isomorphism classes with respect to the homotopy of the morphism between X and Y in the ∞-category $\mathscr{C}$.

Given a morphism f in $\mathscr{C}$ we will denote by $[f]$ its homotopy class in $\mathrm{h}\mathscr{C}$.

Exercise 2.2.10. (i) Check that the above construction is well defined. Namely, that the composition of morphisms descends to the homotopy classes.

(ii) Prove that there exists a canonical map $u : \mathscr{C} \to \mathrm{N}_\bullet\, \mathrm{h}\mathscr{C}$ with the property that for any ordinary category D the induced map

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}\mathscr{C}, \mathsf{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathscr{C}, \mathrm{N}_\bullet(\mathsf{D}))$$

is a bijection.

[Hint: see [34, Tag 0049].]

Example 2.2.11 (a.).

For X a classical scheme and $\mathrm{QCoh}(X)$ (see **??** below) the ∞-category of quasi-coherent sheaves on X, one has $\mathrm{h}\,\mathrm{QCoh}(X) \simeq D(X)$, where $D(X)$ is the ordinary derived category of complexes of quasi-coherent sheaves.

For Spc the category of Definition 2.2.52 one has $\mathrm{h}\,\mathrm{Spc}$ is the homotopy category of topological spaces as considered in [].

For D an ordinary category, it follows from Exercise 2.2.10 that $\mathrm{h}\,\mathrm{N}_\bullet \mathsf{D} \simeq \mathsf{D}$.

For X a topological space, one has

$$\mathrm{h}\,\mathrm{Sing}_\bullet X \simeq \text{fundamental groupoid of } X.$$

Remark 2.2.12. The nerve functor $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$ preserves limits so it admits a left adjoint $\mathrm{h}' : \mathrm{Set}_\Delta \to \mathrm{Cat}$, when a simplicial set $S_\bullet$ correspods to an ∞-category $\mathscr{C}$, one has an equivalence

$$\mathrm{h}'(S_\bullet) \simeq \mathrm{h}\mathscr{C}.$$

Interestingly, since for any ordinary category D its nerve $\mathrm{N}_\bullet\mathsf{D}$ is 2-coskeletal[17] one has an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}\mathscr{C},\mathsf{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta^{\leq 2}}(\mathscr{C}^{\leq 2},\mathrm{N}_\bullet(\mathsf{D})^{\leq 2}),$$

where $\mathrm{Set}_\Delta^{\leq 2} := \mathrm{Fun}((\Delta^{\leq 2})^{\mathrm{op}},\mathrm{Sets})$ and $\mathscr{C}^{\leq 2}$ denotes the restriction of $\mathscr{C}$ to $(\Delta^{\leq 2})^{\mathrm{op}}$. In other words, two simplicial sets that agree on their values on n-simplices for $n \leq 2$ will produce equivalent categories under $\mathrm{h}'(-)$.

Definition 2.2.13. A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be an *isomorphism* if $[f]$ is an isomorphism in the homotopy category of $\mathscr{C}$. Equivalently, f has a left and right homotopy inverses, i.e. there are maps $g, h : Y \to X$ such that

$$g \circ f \sim \mathrm{id}_X \quad \text{and} \quad f \circ h \sim \mathrm{id}_Y.$$

The following is a type of sanity check on the definition of isomorphisms.

Remark 2.2.14. Given an ∞-category $\mathscr{C}$ then $\mathscr{C}$ is a *Kan complex* if and only if every morphism in $\mathscr{C}$ is an isomorphism (see [34, Tag 0052] and [34, Tag 019D]).

2.2.2 Mapping spaces

When studying category theory, we often consider the set of morphisms between two objects. For an ∞-category the morphisms between two fixed objects should be seem as a topological space, i.e. a Kan complex. The next results show that in the model of quasi-categories it is very easy to get a hand on these objects.

Definition-Proposition 2.2.15 ([34, Tag 01JC])**.** Given two objects X and Y in an ∞-category $\mathscr{C}$ we define their mapping space to be the simplicial set

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) := \{X\} \underset{\mathrm{Fun}(\{0\},\mathscr{C})}{\times} \mathrm{Fun}(\Delta^1,\mathscr{C}) \underset{\mathrm{Fun}(\{1\},\mathscr{C})}{\times} \{Y\}.$$

The simplicial set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is a Kan complex.

Warning 2.2.16. Some references (e.g. [16] and [33] on the initial Definition 1.2.2.1, though later the same convention as ours is used.) use the notation $\mathrm{Map}_{\mathscr{C}}(X,Y)$ or $\mathrm{Maps}_{\mathscr{C}}(X,Y)$ for what we denoted by $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ and reserve the notation $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ for the set of morphisms in the homotopy category, which we denoted by $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X,Y)$.

Remark 2.2.17. See [34, Tag 01J3] for a discussion of mapping spaces. Specially the description of the left-pinched and right-pinched space of morphisms.

We want to discuss how one can define the composition of morphisms in an ∞-category. Before that we need to review the homotopy category of spaces.

Definition 2.2.18. The homotopy category of spaces is the ordinary category Top, it can be characterized[18] in either of the following ways:

- $\mathsf{Top} := \mathrm{h}\,\mathrm{Spc}$, i.e. it is the homotopy category of the ∞-category of spaces;
- $\mathsf{Top} := \mathrm{h}\,\mathrm{Kan}$, i.e. Kan is the subcategory of Set_Δ which satisfy the lifting properties for all horns and morphisms are homotopy classes of morphisms[19];
- $\mathsf{Top} := \mathrm{Set}_\Delta[W_{\mathrm{w.h.e.}}]$, i.e. the localization[20] of the category of Set_Δ with respect to the class W_{whe} of weak homotopy equivalences;

[17] I.e. the functor $N_\bullet\mathsf{D} : \Delta^{\mathrm{op}} \to \mathrm{Sets}$ is equivalent to the *right* Kan extension of its restriction to the subcategory $(\Delta^{\leq 2})^{\mathrm{op}}$ generated by $[n]$ for $n \leq 2$.

[18] Or we could say defined, though some of the statements might be circular.

[19] Notice that given two vertices X and Y in a Kan complex K Definition 2.2.5 makes sense for $\mathrm{hom}(X,Y)$ the set of edges of K with domain X and codomain Y.

[20] See find a reference for this.

- Top is the homotopy category associated to the category Set_Δ with the standard model structure, i.e. fibrant objects are Kan complexes and weak equivalences are weak homotopy equivalences;
- Top can also be defined by considering the category of topological spaces which have the homotopy type of a CW complex localized at weak homotopy equivalences (see [34, Tag 012Z]).
- Are there more definitions?

Remark 2.2.19. One of the main insights in setting up the theory of ∞-categories is that the homs between any two objects of an ∞-category only make sense as an object of the category Top, i.e. only the image of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ in the category Top is meaningful. In particular, the underlying set of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is *not* meaningful and one should avoid arguments that rely on the underlying set of the simplicial set that represents this object in the world of quasi-categories.

Proposition 2.2.20. *Let X, Y and Z be objects of an ∞-category $\mathscr{C}$, then there exists a morphism*

$$\circ : Hom_{\mathscr{C}}(Y, Z) \times Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{C}}(X, Y)$$

in the category Top.

Proof. The proof exploits how fibrations are essential in working with higher categories. Consider the simplicial set

$$\mathrm{Hom}_{\mathscr{C}}(X, Y, Z) := \mathrm{Fun}(\Delta^2, \mathscr{C}) \underset{\mathrm{Fun}(\{0,1,2\})}{\times} \{X, Y, Z\}.$$

Pre-composition with the inclusions $\Delta^1 \hookrightarrow \Delta^2$ corresponding to the edges $1 \to 2$ and $0 \to 1$ defines a projection

$$\theta : \mathrm{Hom}_{\mathscr{C}}(X, Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y).$$

The key claim is that p is a trivial Kan fibration, i.e. it has the right lifting property with respect to the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n \geq 0$. We refer the reader to [34, Tag 01PK], take $\mathscr{D}$ to be a point, for a proof. This implies that $[\theta]$ is invertible in the category $\mathrm{h}\,\mathscr{S}\mathrm{pc}$. Thus, the composition is defined as the composite:

$$\mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \overset{[\theta]^{-1}}{\to} \mathrm{Hom}_{\mathscr{C}}(X, Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$$

in the category $\mathrm{h}\,\mathscr{S}\mathrm{pc}$, where the second map is the pre-composition with $\Delta^1 \hookrightarrow \Delta^2$ corresponding to the inclusion of the edge $0 \to 2$. ☐

Remark 2.2.21. Let $\mathscr{C}$ be an ∞-category and $f : X \to Y$ and $g : Y \to Z$ two morphisms, we know that there exists $\sigma : \Delta^2 \to \mathscr{C}$ which witness their composition and that the composite determines an unique morphism in the homotopy category. However, one can say something stronger–the space of σ's witnessing the composition of g and f is contractible. See [34, Tag 007A] for how to formalize and prove this result.

Remark 2.2.22. By Remark 2.2.21 one has that the morphism in 2.2.20 is unique up to a contractible space of choices. In fact, as in usual category theory many notions are well-defined up to an unique isomorphism [21], in ∞-category theory the analogous correct requirement of unicity is that certain piece of data is unique up to a *contractible space of choices.*

2.2.3 Functors

Definition 2.2.23. An *functor $f : \mathscr{C} \to \mathscr{D}$ between ∞-categories* is a map of simplicial sets $f : \mathscr{C} \to \mathscr{D}$. Notice that the set of maps between two simplicial sets assemble into a simplicial set we will denote it by $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$.

Exercise 2.2.24. (i) Given two ordinary categories C and D, one has a bijection:

$$\{\text{ functors } f : \mathsf{C} \to \mathsf{D}\} \simeq \{\text{ functors } f : \mathrm{N}_\bullet\mathsf{C} \to \mathrm{N}_\bullet\mathsf{D}\}\,;$$

[21] In the appropriate ambient category.

(ii) Given $\mathscr{C}$ an ∞-category and D an ordinary category one has a bijection:

$$\{\text{ functors } f : \mathscr{C} \to \mathrm{N}_\bullet \mathsf{D}\} \simeq \{\text{ functors } f : \mathrm{h}\mathscr{C} \to \mathsf{D}\}$$

(iii) For $\mathscr{C}$ an ∞-category and X a topological space one has a bijection:

$$\{\text{ functors } f : \mathscr{C} \to \mathrm{Sing}_\bullet(X)\} \simeq \{\text{ continuous functions } f : |\mathscr{C}| \to X\}^{22}.$$

Remark 2.2.25. It is interesting to note that a functor between ∞-categories carries a lot of data. Let f, g be composable morphisms in an ∞-category $\mathscr{C}$ and h one composition of g and f. A functor $F : \mathscr{C} \to \mathscr{D}$ to an ∞-category $\mathscr{D}$ not only sends h to a composition of $F(g)$ and $F(f)$ it a *witness* of the composition h, i.e. a 2-simplex $\sigma : \Delta^2 \to \mathscr{C}$ whose restriction to $\partial\Delta^2$ is the diagram (2.2) to a 2-simplex $F \circ \sigma : \Delta^2 \to \mathscr{D}$ witnessing the composition of $F(g)$ with $F(f)$. Thus, in general to specific a functor between two arbitrary ∞-categories involves specifying a lot of data.

The following result is important in performing the construction of the ∞-category of functors between two given ∞-categories. Given $S_\bullet$ a simplicial set and $\mathscr{C}$ an ∞-category, let $\mathrm{Fun}(S_\bullet, \mathscr{C})$ denote the simplicial set of maps of simplicial sets between them one has

Proposition 2.2.26 ([34, Tag 0065])**.** *The simplicial set Fun($S_\bullet, \mathscr{C}$) is an ∞-category . In particular, given two ∞-categories $\mathscr{C}$ and $\mathscr{D}$ the simplicial set Fun($\mathscr{C}, \mathscr{D}$) of functors between them is an ∞-category .*

Definition 2.2.27. A functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories is an *equivalence* if it there exists a functor $G : \mathscr{D} \to \mathscr{C}$ such that

$$G \circ F \simeq \mathrm{id}_{\mathscr{C}} \text{ in } \mathrm{Fun}(\mathscr{C}, \mathscr{C}) \text{ and } F \circ G \simeq \mathrm{id}_{\mathscr{D}} \text{ in } \mathrm{Fun}(\mathscr{D}, \mathscr{D}),$$

where we recall that $\mathrm{Fun}(\mathscr{C}, \mathscr{C})$ is seem as an ∞-category and the notion of isomorphic morphisms, e.g. $G \circ F \simeq \mathrm{id}_{\mathscr{C}}$ is as described in 2.2.13.

Remark 2.2.28. Once we define Cat_∞ the ∞-category of ∞-categories (see §2.2.6 below). Definition 2.2.27 can be rephrased more conceptually as $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of ∞-categories if it is an equivalence as a morphism in the ∞-category Cat_∞, i.e. it induces an equivalence on $\mathrm{h}(\mathrm{Cat}_\infty)$.

The main result we want to discuss in this section is that the usual characterization of equivalence of categories from ordinary category theory, i.e. functors which are fully faithful and essentially surjective, also makes sense in ∞-category theory. First we need a couple of definitions.

Definition 2.2.29. A functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories $\mathscr{C}$ and $\mathscr{D}$ is said to be

- *fully faithful* if for every pair of objects X, Y in $\mathscr{C}$ the map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$$

  is a homotopy equivalence of Kan complexes;

- *essentially surjective* if for every object Y in $\mathscr{D}$ one has an object X in $\mathscr{C}$ such that

$$F(X) \simeq Y,$$

  equivalently F induces an equivalence $\pi_0(\mathscr{C}^{\simeq}) \simeq \pi_0\pi_0(\mathscr{D}^{\simeq})$.

Theorem 2.2.30 ([34, Tag 01JX])**.** *A functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of ∞-categories if and only if F is fully faithful and essentially surjective.*

We finish this subsection with a picture that might help contextualize ∞-categories:

$$\begin{array}{ccccc}
\{\text{ 1 -- categories }\} & \xhookrightarrow{N_\bullet} & \{\text{ } \infty\text{-categories }\} & \hookleftarrow & \{\text{ Kan complexes }\} \\
 & & \big\downarrow & & \big\uparrow \\
 & & \{\text{ simplicial sets }\} & & \{\text{ topological space }\}
\end{array}$$

[22]Here $|\mathscr{C}|$ denotes the geometric realization of the corresponding simplicial set.

2.2.4 Commutative diagrams

We now embark on a small digression about commutative diagrams in an ∞-category , for more on this topic see [34, Tag 005H] and [33, §1.2.6].

Definition 2.2.31. Let G denote a graph with no loops and an unique edge between any two of its vertices. A *G-commutative diagram* in an ∞-category $\mathscr{C}$ is a map of simplicial sets $\sigma : \mathrm{N}_\bullet G \to \mathscr{C}$, where we see G as a 1-category.

Remark 2.2.32. Let C be an ordinary category which we regard as an ∞-category by taking its nerve. Notice that we can then unambiguously see G as an ordinary category. There are then two notions that we could consider as a G-commutative diagram in C:

- a map of 1-categories, i.e. a functor, $\sigma : G \to \mathsf{C}$;
- a map of simplicial sets $\sigma : \mathrm{N}_\bullet G \to \mathrm{N}_\bullet \mathsf{C}$.

Fortunately, there is no risk of confusion, since by [34, Tag 005P] these data are equivalent.

Warning 2.2.33. Remark 2.2.32 does *not* generalize to ∞-categories.

To appreciate what is happening let's consider a simple but instructive example of the equivalence in Remark 2.2.32.

Example 2.2.34. Consider the graph $I = [1] \times [1]$ and the simplicial set $\Delta^1 \times \Delta^1 = \mathrm{N}_\bullet([1] \times [1])$. Notice this simplicial set contains

- four vertices corresponding to the vertices of I;
- five edges, four correspond to the edges of I and an extra 1-simplex given by the composition of upper and right edges, equivalently left and lower edges;
- two 2-simplices corresponding to the witnesses of the composition of the commutative of the upper or lower triangle.

Let $K_\bullet = \partial(\Delta^1 \times \Delta^1)$ denote the boundary of the simplicial set $\Delta^1 \times \Delta^1$. Then, Remark 2.2.32 says that the data of $\sigma : \Delta^1 \times \Delta^1 \to \mathrm{N}_\bullet \mathsf{C}$ is equivalent to the usual notion of a commutative square in the ordinary category C. This however fails for a general ∞-category $\mathscr{C}$. Can I find an explicit example?

Remark 2.2.35. Example 2.2.34 illustrates the dichotomy between the behavior of a commutative square in an ordinary category, which is a *property* that one can check versus a commutative square in an ∞-category which involves extra *data*. An even when one can prove that such extra data exists seldom it won't involve any choices (cf. Exercise 2.2.36).

The following exercise shows that there is a silver lining in this very special example.

Exercise 2.2.36. Let $\sigma : \partial(\Delta^1 \times \Delta^1) \to \mathscr{C}$ be a map of simplicial sets[23], i.e. the following data

$$\begin{array}{ccc} X_{00} & \xrightarrow{f} & X_{01} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X_{10} & \xrightarrow[f']{} & X_{11} \end{array} \tag{2.3}$$

Consider the diagram $\overline{\sigma} : \partial(\Delta^1 \times \Delta^1) \to \mathrm{N}_\bullet\, \mathrm{h}\mathscr{C}$ obtained by composing with the canonical map $\mathscr{C} \to \mathrm{N}_\bullet\, \mathrm{h}\mathscr{C}$, i.e. a commutative diagram

$$\begin{array}{ccc} X_{00} & \xrightarrow{[f]} & X_{01} \\ {\scriptstyle [g]}\downarrow & & \downarrow{\scriptstyle [g']} \\ X_{10} & \xrightarrow[{[f']}]{} & X_{11} \end{array} \tag{2.4}$$

Prove that (2.4) is commutative if and only if (2.3), i.e. the map σ extends to a map $\tilde{\sigma} : \Delta^1 \times \Delta^1 \to \mathscr{C}$. Notice that this extension is not unique.

[23] Notice this is weaker than the data of an I-commutative diagram in $\mathscr{C}$.

Remark 2.2.37. The problem of lifting the data of a commutative diagram in the homotopy category to a commutative diagram in the original ∞-category is sometimes referred to a homotopy coherence problem. We refer the reader to [33, §1.2.6] and the second example in [38, §5] for interesting discussions of this.

2.2.5 Examples of ∞-categories

In this section we mention three variations on the nerve construction which can take in a category with more structure, i.e. enriched over some symmetric monoidal category, and produce an ∞-category .

Homotopy coherent nerve

As we mentioned in §2.1.2 another possible model for ∞-categories is the theory of categories enriched in simplicial sets, sometimes simply called simplicial categories.

The homotopy coherent nerve associates an ∞-category (i.e. quasi-category) to a simplicial category satisfying a certain condition. To describe this construction we need to introduce a sort of resolution of the category $[n]$ as a simplicial category.

Definition 2.2.38. For any $n \geq 0$ we let $\mathrm{Path}([n])$ denote the simplicial category whose

- objects are the same as the objects of $[n]$;
- for x, y objects of $[n]$ we define

$$\mathrm{Hom}_{\mathrm{Path}[n]}(x,y) := \mathrm{N}_\bullet\,(\text{ partially ordered set of sequences } x = x_0 < x_1 < \cdots < x_m = y\ )$$

  where the sequences are ordered by *reverse* inclusion.

The identity morphisms of x correspond to the poset $\{x\}$ and composition is given by union of sets and considering the inherited order.

Here is an example to clarify Definition 2.2.38 a bit.

Example 2.2.39. In $\mathrm{Path}([2])$ the simplicial set of morphisms between 0 and 2 is the nerve of the following *noncommutative* diagram

$$\begin{array}{ccc} 0 & \xrightarrow{0,2} & 2 \\ \Big\downarrow{\scriptstyle 0,1} & & \Big\downarrow{\scriptstyle 2} \\ 1 & \xrightarrow{1,2} & 2 \end{array}$$

That is the composite $0,1,2 = 1,2 \circ 0,1$ is *not* equal to $0,2$. However, one has a canonical homotopy between them, thus making the simplicial set $\mathrm{Hom}_{\mathrm{Path}[2]}(0,2)$ contractible. More generally, the simplicial set of morphisms between two vertices x, y in $\mathrm{Path}([n])$ is a hypercube that keeps track of all possible compositions of morphisms but that doesn't impose equality of this compositions. See [33, Remark 1.1.5.2]. Notice that the category $[2]$ can be seen as a simplicial category with the constant simplicial sets as morphism spaces. One has a natural functor of simplicial categories $\mathrm{Path}([2]) \to [2]$, however this is *not* an equivalence of categories (see [34, Tag 00KX]).

Definition 2.2.40 (Homotopy coherent nerve)**.** Let C be a simplicial category, we define the *homotopy coherent nerve*[24] as

$$\mathrm{N}^{\mathrm{hc}}_\bullet \mathsf{C} := \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathrm{Path}([n]), \mathsf{C}).$$

Proposition 2.2.41 ([34, Tag OOLJ])**.** *Let C be a simplicial category and suppose that for every pair of objects X, Y in C the simplicial set $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ is a Kan complex*[25]*. Then $N^{\mathrm{hc}}_\bullet(\mathsf{C})$ is an ∞-category.*

One way to argue that the homotopy coherent nerve is producing the 'correct' ∞-category is to investigate the relation between the mapping spaces of the resulting ∞-category and those of the initial simplicial category. One has the following result:

[24] In [33] this construction is referred to as the *simplicial nerve* and its notation in *loc. cit.* is identical to the regular nerve.

[25] Notice that this kind of condition is expected since in the theory of ∞-categories the set of morphisms between two objects should naturally be a topological space, i.e. Kan complex.

Proposition 2.2.42. *For any fibrant simplicial category*[26] C *the canonical map*

$$\underline{Hom}_{\mathsf{C}}(X,Y) \to Hom_{N_\bullet^{hc}(\mathsf{C})}(X,Y)$$

is an equivalence in Top.

Example 2.2.43. The homotopy coherent nerve can be used to pass from topological categories to quasi-categories. Indeed, suppose that C is a category enriched in topological spaces, then we define a simplicial category $\mathrm{Sing}(\mathsf{C})$ by considering the space objects and the morphisms are defined by

$$\mathrm{Hom}_{\mathrm{Sing}(\mathsf{C})}(X,Y) := \mathrm{Sing}_\bullet\left(\mathrm{Hom}_{\mathsf{C}}(X,Y)\right).$$

Notice that the above also determines what to do for the composition morphisms, since the functor Sing preserves finite limits.

Example 2.2.44. Find an interesting example of simplicial category.

Remark 2.2.45. One has an analogous nerve construction that takes as input a topological category, i.e. a category enriched in the (ordinary) category of topological spaces. See [35, Construction 0.2.2.9] the main difference is that instead of considering the simplicial categories $\mathrm{Path}[n]$ above one considers the topological categories $\mathscr{T}_n$ (see [35, Construction 0.2.2.6]) whose objects are the same as the objects of $[n]$ and morphisms are

$$\mathrm{Hom}_{\mathscr{T}_n}(i,j) = \Big\{\emptyset \ \text{ if } i>j; \{f \in [0,1]^{i,i+1,\dots,j} \mid f(i) = f(j) = 1\} \ \text{ if } i \le j.$$

Differential Graded Nerve

Another way of producing ∞-categories is to start with a differential graded category (dg-category). We recall that a *dg-category* is a 1-category enriched in complexes of abelian groups. More concretely, given C a dg-category for every pair of objects X and Y in C one has a chain complex $\mathrm{Hom}_{\mathsf{C}}(X,Y)_\bullet$ and for every triple X, Y and Z in C one has a composition map

$$C_{Z,Y,X} : \mathrm{Hom}_{\mathsf{C}}(Y,Z)_n \times \mathrm{Hom}_{\mathsf{C}}(X,Y)_m \to \mathrm{Hom}_{\mathsf{C}}(X,Z)_{n+m}$$

for every $n, m \in \mathbb{Z}$.

Definition 2.2.46. Given a dg-category C, we let $\mathrm{N}^{\mathrm{dg}}(\mathsf{C})_n$ denote the collection of ordered pairs $(\{X_i\}_{0\le i\le n}, f_I)$ where:

- each X_i is an object of C;
- $I = \{i_0 > i_1 > \cdots > i_k\} \subseteq [n]$ is a subset with at least two elements and $f_I \in \mathrm{Hom}_{\mathsf{C}}(X_{i_k}, X_{i_0})_{k-1}$ are morphisms satisfying the identity

$$df_I = \sum_{a=1}^{k-1}(-1)^a (f_{\{i_0>\cdots>i_a\}} \circ f_{\{i_a>\cdots>i_k\}} - f_{I\setminus\{i_a\}}).$$

Given a morphism $\alpha : [n] \to [m]$ in Δ, i.e. a non-decreasing function we define a morphism of $\alpha^* : N_m^{\mathrm{dg}}(\mathsf{C}) \to N_n^{\mathrm{dg}}(\mathsf{C})$ via

$$(\{X_i\}_{0\le i\le m}, f_I) \to ((\{X_i\}_{0\le i\le n}, g_J)),$$

$$g_J = \begin{cases} f_{\alpha(J)} \text{ if } \ \alpha|_J \ \text{ is injective} \\ \mathrm{id}_{X_i} \text{ if } J = \{j_0 > j_1\} \text{ with } \alpha(j_0) = i = \alpha(j_1) \\ 0 \text{ otherwise.} \end{cases}$$

We leave it to the reader to check that these α^* are well-defined and that they assemble into a simplicial set $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathsf{C})$, which we call the *differential graded (dg) nerve* of C.

[26] I.e. satisfying the condition that any simplicial set of morphisms between objects is actually a Kan complex

Remark 2.2.47. Notice that the lower dimensional simplices of $N^{dg}_\nu(\mathsf{C})$ are easy to describe:

- $N^{dg}_0(\mathsf{C})$ is the collection of objects of C;
- $N^{dg}_1(\mathsf{C})$ is the data of a triple $(\{X_0, X_1\}, f \in \mathrm{Hom}_{\mathsf{C}}(X_0, X_1)_0)$, where f is a 0-cycle, i.e. $df = 0$;
- $N^{dg}_2(\mathsf{C})$ is the data of $(\{X_0, X_1, X_2\}, \{f_{01} \in \mathrm{Hom}_{\mathsf{C}}(X_0, X_1)_0, f_{12} \in \mathrm{Hom}_{\mathsf{C}}(X_1, X_2)_0, f_{02} \in \mathrm{Hom}_{\mathsf{C}}(X_0, X_2)_0, f_{012} \in \mathrm{Hom}_{\mathsf{C}}(X_0, X_2)_1\})$, where f_{01}, f_{12} and f_{02} are 0-cycles and satisfy the equation:

$$d(f_{012}) = f_{02} - f_{12} \circ f_{01}.$$

Proposition 2.2.48 ([34, Tag 00PW])**.** *For any dg-category* C *the dg-nerve* $N^{dg}_\bullet(\mathsf{C})$ *is an* ∞*-category .*

Example 2.2.49. Given A an abelian category with enough projective objects the dg-nerve is convenient to construct the derived ∞-category $\mathscr{D}^-(\mathsf{A})$ corresponding to A. Let $\mathsf{A}_{\mathrm{proj}}$ denote the subcategory of A spanned by its projective objects, consider $\mathrm{Ch}^-(\mathsf{A}_{\mathrm{proj}})$ the category of cochain complexes which vanish for sufficiently positive degree. Then one defines

$$\mathscr{D}^-(\mathsf{A}) := N^{dg}_\bullet\left(\mathrm{Ch}^-(\mathsf{A}_{\mathrm{proj}})\right).$$

The homotopy category $\mathrm{h}\,\mathscr{D}^-(\mathsf{A})$ is the "usual" bounded above derived category of A studied in classical homological algebra[27]. Moreover, we will later define the notion of a stable ∞-category, by [32, Corollary 1.3.2.18] the category $\mathscr{D}^-(\mathsf{A})$ is stable. In fact, with the theory of t-structure on stable ∞-categories one can characterize the universal property of $\mathscr{D}^-(\mathsf{A})$.

Example 2.2.50. A particularly important special case of Example 2.2.49 is that when k is a field, then we have

$$\mathrm{Vect}_k^- := N^{dg}_\bullet(\mathrm{Ch}^-(k)),$$

where $\mathrm{Ch}^-(k)$ means the abelian category of bounded above cochain complexes of k-vector spaces. We will sometimes abuse terminology and refer to Vect_k^- as the ∞-category of vector spaces over k. We notice that to obtain the ∞-category whose homotopy category corresponds to the derived category of unbounded complexes over k one needs to take the right completion of Vect_k^- with respect to its t-structure, i.e.

$$\mathrm{Vect} := \lim_{n\geq 0}\left(\cdots \mathrm{Vect}^{\leq(n+1)} \overset{\tau^{\leq n}}{\to} \mathrm{Vect}^{\leq n}\cdots\right).$$

Duskin nerve (see [34, Tag 009T])

Let $\mathbb{C}$ be a 2-category in the sense of Bénabou [], informally speaking $\mathbb{C}$ is a category enriched in categories. Suppose moreover that for any two objects X and Y in $\mathbb{C}$ the category

$$\mathrm{Hom}_{\mathbb{C}}(X, Y)$$

is actually a groupoid. That is to say that $\mathbb{C}$ is a $(2,1)$-category, since every 2-morphism in $\mathbb{C}$ is invertible. Then the so-called Duskin nerve construction $N^{D}_\bullet(\mathbb{C})$ produces an ∞-category . Moreover, this assignment gives a fully faithful functor from the 1-category whose objects are 2-categories and morphisms are unitary lax functors into the category of ∞-categories[28].

Remark 2.2.51. The above three nerve constructions are compatible with the ordinary nerve we described in §2.2.1. More precisely, suppose given C an ordinary category, we can then see C as a 2-category with only identity morphisms or as a simplicial category with the constant simplicial set as its space of morphisms, or yet as a dg category with all morphisms concentrated in degree 0, then one has equivalences of ∞-categories (see [34, Tag 00KZ] and [34, Tag 00PU]):

$$N_\bullet(\mathsf{C}) \simeq N^{D}_\bullet(\mathsf{C}) \simeq N^{hc}_\bullet(\mathsf{C}) \simeq N^{dg}_\bullet(\mathsf{C})$$

[27] The homotopy category $\mathrm{h}\,\mathscr{D}^-(\mathsf{A})$ can be concretely described as considering bounded above cochain complexes of projective modules and localizing it with respect to chain homotopy equivalence. That this is equivalent to starting with all complexes in A bounded above and inverting quasi-isomorphisms is proved for instance in Theorem 10.4.8 in [59].

[28] See [34, Tag 00AU] for the proof of this result and [34, Tag 008G] for the notion of unitary lax functors.

2.2.6 The ∞-categories of spaces and of ∞-categories

The ∞-category of spaces Spc plays the same role in the theory of ∞-categories that the category of sets plays in the usual category theory of 1-categories. Even more importantly, because of the inherent difficulties to work homotopy-coherently all the time when performing manipulations with ∞-categories one relies in the good properties of Spc even more. In this section we present a straightforward construction of Spc as a quasi-category and mention a couple of useful properties.

Construction of the ∞-category Spc A simple and quick construction of the ∞-category $\mathscr{S}$pc is as follows, let Kan $\subseteq \mathrm{Set}_\Delta$ denote the subcategory of simplicial sets which are Kan complexes. Since Set_Δ is a simplicial category, i.e. given two simplicial sets K and S their simplicial mapping space $\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, S)_\bullet$ is defined by

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, S)_n := \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times K, S).$$

The subcategory Kan is a simplicial category, moreover it is not hard to check that if K and S are Kan complexes then so is $\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, S)_\bullet$. So we have

Definition 2.2.52 (∞-category of spaces)**.** We ∞-category of spaces is defined as

$$\mathrm{Spc} := \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}).$$

Remark 2.2.53. Any other model of Spc that is equivalent to it as an ∞-category would be good enough for our purposes.

Remark 2.2.54. In particular, one can consider the simplicial localization of the category Set_Δ with respect to weak homotopy equivalences, i.e. the so-called Dwyer–Kan localization produces a simplicial category from a category and a collection of morphisms in it. The associated ∞-category, obtained by applying the homotopy coherent nerve is equivalent to the more straightforward construction above (Insert reference here.).

Proposition 2.2.55. *The category Spc admits all (small) limits and colimits.*

Proof. The main point is that any ∞-category of presheaves $\mathscr{P}(S) := \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Spc})$ on a simplicial set admits small limits and colimits. See [33, §5.1.2]. It is circular to reason like this for Spc. ☐

Proposition 2.2.56. *The category Spc is equivalent to the cocompletion, i.e. formally adjoint all colimits, of the ∞-category with a single object.*

There is another way to characterize the objects of Spc.

Definition 2.2.57. One says that an ∞-category $\mathscr{C}$ is an *∞-groupoid* if all of its morphisms are invertible, i.e. if its homotopy category $\mathrm{h}\mathscr{C}$ is a groupoid.

Then one has the following sanity check

Proposition 2.2.58. *Given a quasi-category $S_\bullet$ representing an ∞-category $\mathscr{C}$ the following are equivalent:*

(i) $S_\bullet$ is a Kan complex;

(ii) $\mathscr{C}$ is an ∞-groupoid.

Proof. See [33, Proposition 1.2.5.3]. ☐

What else should one mention about this category?

Construction of the ∞-category Cat_∞ We define a simplicial category Cat_∞ as follows:

- objects of Cat_∞ are (small) ∞-categories;
- given two ∞-categories $\mathscr{C}$ and $\mathscr{D}$ we let
$$\mathrm{Hom}_{\mathsf{Cat}_\infty}(\mathscr{C},\mathscr{D}) := (\mathrm{Fun}(\mathscr{C},\mathscr{D}))^{\simeq},$$
i.e. the largest Kan complex inside the ∞-category $\mathrm{Fun}(\mathscr{C},\mathscr{D})$[29].

Definition 2.2.59 (∞-category of ∞-categories)**.** We define the *∞-category of ∞-categories* as
$$\mathrm{Cat}_\infty := \mathrm{N}^{\mathrm{hc}}_\bullet(\mathsf{Cat}_\infty),$$
i.e. the homotopy coherent nerve of the simplicial category Cat_∞.

Remark 2.2.60. Notice that the last step of taking the homotopy coherent nerve is necessary to produce an ∞-category in our preferred model of quasi-categories. From a model-independent point of view the first construction is already good enough.

Remark 2.2.61. Equivalently, we can define the ∞-category Cat_∞ by taking the simplicial localization of the category of simplicial sets with respect to Dwyer–Kan equivalences. (Find a reference.)

Remark 2.2.62. More importantly, one would like to construct the ∞-category of ∞-categories from a simplicial model structure[30]. It is however not enough to consider the Joyal model structure[31] on simplicial sets for this purpose, since this model structure is *not* compatible with the simplicial enrichment. The solution found by Lurie in [33, Chapter 3] is to consider the category of marked simplicial sets Set^+_Δ. A *marked simplicial set* is a pair $(S,\mathscr{E}_S)$ of a simplicial set S and a subset $\mathscr{E}_S$ of its edges containing the degenerate ones and morphisms are required to preserve marked edges. In particular, any ∞-category $\mathscr{C}$ determines a marked simplicial set $\mathscr{C}^\natural$ by marking its equivalences. The main technical input then is to show that the category Set^+_Δ admits a simplicial model structure whose cofibrant-fibrant objects are the simplicial sets of the form $\mathscr{C}^\natural$ for some ∞-category $\mathscr{C}$. Then one has an alternative presentation of Cat_∞ as
$$\mathrm{Cat}_\infty \simeq \mathrm{N}^{\mathrm{hc}}_\bullet\left(\mathbf{Set}^+_{\Delta,\mathrm{cf}}\right),$$
where $\mathbf{Set}^+_{\Delta,\mathrm{cf}}$ denotes the subcategory of fibrant and cofibrant objects.

Remark 2.2.63. In particular, the technique of marked simplicial sets allows one to associate an ∞-category to an *arbitrary* model category M. Indeed, given M we consider the pair (M,W) where W denotes the class of weak equivalences of M, i.e. we forget its fibrations and cofibrations. An enhanced version of the *ordinary* nerve construction associates to this data $\mathrm{N}_\bullet(\mathsf{M},W)$ a marked simplicial set. One then obtains an ∞-category by taking a fibrant and cofibrant replacement of $\mathrm{N}_\bullet(\mathsf{M},W)$ with respect to the model structure on marked simplicial sets.

2.3 Higher Category Theory

If one opens any book in usual category theory, here is an incomplete list of important notions of the theory that one might want to make sense in the framework of ∞-categories:

(i) initial and final objects;

(ii) limits and colimts;

(iii) adjoint functors;

(iv) the Yoneda lemma.

Let's try to formulate these concepts in the language of ∞-categories and see where things get complicated.

[29] This condition is to guarantee that the homotopy coherent nerve of this simplicial category is a quasi-category. In analogy to ordinary category theory this is restricting to natural transformations which are equivalences so that one has 1-*category* of categories, inside of a proper 2-category.

[30] One of the reasons for that is that this would formally imply the existence of limits and colimits of ∞-categories, from the existence of homotopy limits and colimits of this model category.

[31] The one where the quasi-categories are the fibrant objects.

2.3.1 Initial and final objects

Given an ∞-category $\mathscr{C}$, one might pose the following naïve definition:

Definition 2.3.1. An object $X \in \mathscr{C}$ is

- *initial* if for any object $Y \in \mathscr{C}$ one has
$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \simeq \mathrm{pt} \quad \text{in Spc;}$$
- *final* if for any object $Y \in \mathscr{C}$ one has
$$\mathrm{Hom}_{\mathscr{C}}(Y, X) \simeq \mathrm{pt} \quad \text{in Spc.}$$

Definition 2.3.1 turns out to capture the correct concept. Here is a little justification.

Lemma 2.3.2. *Let $\mathscr{C}$ be an ∞-category that admits an initial (resp. final) object, then the space of initial (resp. final) objects of $\mathscr{C}$ is contractible.*

Proof. Make sense and prove this statement. □

2.3.2 Limits and colimits

Consider a functor $F : \mathsf{K} \to \mathsf{C}$ between two ordinary categories. Here is a sleek way of defining what the colimit of F is. Consider $\Delta_F : \mathsf{C} \to \mathrm{Fun}(\mathsf{K}, \mathsf{C})$ the functor that sends any object $X \in \mathsf{C}$ to the constant diagram $c_X : \mathsf{K} \to \mathsf{C}$[32] the *colimit of* F is an object colim $F \in \mathsf{C}$ which co-represents the functor

$$\begin{aligned} \mathsf{C} &\to \mathrm{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathrm{Fun}(\mathsf{K},\mathsf{C})}(F, \Delta_F(X)). \end{aligned} \tag{2.5}$$

Explicitly, the data of colim F is determined by isomorphisms

$$\mathrm{Hom}_{\mathrm{Fun}(\mathsf{K},\mathsf{C})}(F, \Delta_F(X)) \simeq \mathrm{Hom}_{\mathsf{C}}(\mathrm{colim}\, F, X) \tag{2.6}$$

for all $X \in \mathsf{C}$. Moreover, the isomorphisms (2.6) need to be functorial in X.

Now suppose that one has a functor $F : \mathscr{K} \to \mathscr{C}$ between ∞-categories $\mathscr{K}$ and $\mathscr{C}$. If one tries to naïvely mimic the above definition of colimit for ∞-categories, two problems arise:

Problem 2.3.3. What functor between ∞-categories should play the role of (2.5)? Clearly, one needs a functor
$$\mathrm{Hom}_{\mathscr{C}} : \mathscr{C} \to \mathrm{Spc}$$
from $\mathscr{C}$ to the ∞-category of spaces, but how does one *write* such a functor?

Problem 2.3.4. In the right-hand side of (2.6) the 1-morphisms in the category $\mathrm{Fun}(\mathscr{K}, \mathscr{C})$ are not necessarily invertible, in other words, $\mathrm{Fun}(\mathscr{K}, \mathscr{C})$ here *can not* be considered as the morphisms between $\mathscr{K}$ and $\mathscr{C}$ seen as objects of the ∞-category Cat_∞.

Let's try again to define colimits, this time using the notion that we already have from §2.3.1. Recall from classical category theory that given a functor $F : \mathsf{K} \to \mathsf{C}$ one can define a slice category $\mathsf{C}^{F/}$ whose

- objects are elements $X \in \mathsf{C}$ together with a morphism from every vertex $F(k)$, for $k \in \mathsf{K}$ which are compatible with the image of the morphisms in K;
- morphisms are maps $f : X \to Y$ in C which are compatible with all the data encoded in X and Y being objects of $\mathscr{C}^{F/}$.

one can also define the colimit of a functor $F : \mathsf{K} \to \mathsf{C}$ as the

[32] I.e. the value of c_X on every vertex of K is X and on each morphism is the identity morphisms id_X.

Definition 2.3.5. Given a functor $F : \mathsf{K} \to \mathsf{C}$ between ordinary categories a *colimit of* F is an initial object in the category $\mathsf{C}^{F/}$.

Remark 2.3.6. As usual we will abuse of notation and simply refer to the object colim $F \in \mathsf{C}$ corresponding to the image of the colimit of F under the map $\mathscr{C}^{F/-} \to \mathsf{C}$ as the colimit of F.

Definition 2.3.5 gives us an alternative to defining colimits in ∞-categories. Since we already have a notion of initial objects all we need is to define the slice category corresponding to a functor between ∞-categories.

As a warm-up we start with the case of a functor $X : [0] \to \mathscr{C}$, i.e. just an object $X \in \mathscr{C}$.

Definition 2.3.7. Let $X \in \mathscr{C}$ be an object of a category $\mathscr{C}$ we define the over and under slice categories for X as follows:[33]

- $\mathscr{C}_{X/}$ is the pullback[34]

$$\begin{array}{ccc} \mathscr{C}_{X/} & \longrightarrow & \mathrm{Fun}([1],\mathscr{C}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathscr{C} & \xrightarrow{\{X\}\times \mathrm{id}_{\mathscr{C}}} & \mathrm{Fun}(\{0,1\},\mathscr{C}) \end{array}$$

- $\mathscr{C}_{/X}$ is the pullback

$$\begin{array}{ccc} \mathscr{C}_{/X} & \longrightarrow & \mathrm{Fun}([1],\mathscr{C}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathscr{C} & \xrightarrow{\mathrm{id}_{\mathscr{C}}\times\{X\}} & \mathrm{Fun}(\{0,1\},\mathscr{C}) \end{array}$$

More generally[35], given a diagram[36] one can define $\mathscr{C}^{F/}$ by the following diagram where both squares are pullbacks

$$\begin{array}{ccccc} \mathscr{C}^{F/-} & \longrightarrow & \mathrm{Fun}(K,\mathscr{C})^{F/-} & \longrightarrow & \mathrm{Fun}(K\times[1],\mathscr{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow{\Delta_F} & \{F\}\times\mathrm{Fun}(K,\mathscr{C}) & \longrightarrow & \mathrm{Fun}(K\times\{0,1\},\mathscr{C}), \end{array}$$

where $\Delta_F : \mathscr{C} \to \mathrm{Fun}(K,\mathscr{C})$ sends an object $X \in \mathscr{C}$ to the constant functor from K to X. Similarly, one can define $\mathscr{C}^{/F}$ by an analogous diagram.

Proposition 2.3.8. *For any diagram $F : K \to \mathscr{C}$ the simplicial set $\mathscr{C}^{F/}$ is an ∞-category.*

Idea of proof. We will only consider the case $\mathscr{C}^{X/}$. Notice that for any $Y \in \mathscr{C}$ one has a pullback diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(X,Y) & \longrightarrow & \mathscr{C}^{X/-} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \{Y\} & \longrightarrow & \mathscr{C} \end{array}$$

We claim that p is a left fibration (see below). For instance for

$$\begin{array}{ccc} \Lambda^1_0 & \longrightarrow & \mathscr{C}^{X/} \\ \downarrow & & \downarrow \\ \Delta^2 & \longrightarrow & \mathscr{C} \end{array}$$

[33]Notice this doesn't exactly restrict to the definition of what is denoted by $\mathscr{C}_{X/}$ (or $\mathscr{C}_{/X}$) in [33, §1.2.9] for quasi-categories, however see [44, §55] for a proof that $\mathscr{C}^{X/}$ and $\mathscr{C}_{/X}$ agree.

[34]Here we mean the pullback as ∞-categories, which in the model of quasi-categories can be taken simply as the usual pullback of the underlying simplicial sets.

[35]I learned this neat definition from [44, §55.6], we refer the reader to Rezk's notes for more details.

[36]I.e. K is a simplicial set, $\mathscr{C}$ an ∞-category and F a map of simplicial sets.

this is saying that any diagram $X \to Y \to Z$ in $\mathscr{C}$ can be composed with a witness to the composition, which is clear from the definition of $\mathscr{C}$.

Thus, one has that the composite $\mathscr{C}^{X/} \to \mathscr{C} \to \mathrm{pt}$ is an inner fibration, i.e. has the lifting property that defines ∞-categories. Thus, $\mathscr{C}^{X/}$ is an ∞-category. □

2.3.3 Model independent framework

In this section we formulate certain categorical the notions for ∞-categories using what we have already defined in §2.2.1.

Convention 2.3.9. In the following we will refer to $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ to any object in the ∞-category Spc which is equivalent to $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Definition 2.3.10. Let $F : \mathscr{D} \to \mathscr{C}$ be a map between ∞-categories, we will say that $\mathscr{D}$ is a *subcategory* of $\mathscr{C}$ if the induced functor $\mathrm{h}\, F : \mathrm{h}\,\mathscr{D} \to \mathrm{h}\,\mathscr{C}$ between their homotopy categories realizes $\mathrm{h}\,\mathscr{D}$ as a subcategory of $\mathrm{h}\,\mathscr{C}$.

n-truncation of an ∞-category

The following notion is an analogue of the filtration

$$\mathrm{Spc}^{\leq 0} \hookrightarrow \mathrm{Spc}^{\leq 1} \hookrightarrow \cdots \hookrightarrow \mathrm{Spc}^{\leq n} \hookrightarrow \cdots \hookrightarrow \mathrm{Spc}$$

for an arbitrary ∞-category $\mathscr{C}$.

Definition 2.3.11. Let $X \in \mathscr{C}$ be an object of $\mathscr{C}$, we say that X is n-truncated if for every $A \in \mathscr{C}$ one has

$$\mathrm{Hom}_{\mathscr{C}}(A, X) \in \mathrm{Spc}^{\leq n},$$

i.e. $\pi_k(\mathrm{Hom}_{\mathscr{C}}(A, X)) = 0$ for all $k > n$. We let $\tau^{\leq n}\mathscr{C}$ denote the full subcategory of $\mathscr{C}$ generated by n-truncated objects. In particular, one calls $\tau^{\leq 0}\mathscr{C}$ the subcategory of *discrete* objects of $\mathscr{C}$.

Here are a couple of properties of truncated objects:

Proposition 2.3.12. *(i) For any $n \geq 0$ the subcategory $\tau^{\leq n}\mathscr{C} \hookrightarrow \mathscr{C}$ is stable under all limits which exists in $\mathscr{C}$.*

(ii) Any functor that preserves finite limits will preserves n-truncated objects and morphisms[37].

(iii) When $\mathscr{C}$ is presentable[38] *then the inclusion $\tau^{\leq n}\mathscr{C} \hookrightarrow \mathscr{C}$ admits a left adjoint*

$$\tau^{\leq n} : \mathscr{C} \to \tau^{\leq n}\mathscr{C}.$$

Explain the informal way of working model independently.

2.3.4 Limits, colimits and Kan extensions

From this section forward we change our conventions. From now on we will only say category for an ∞-category. We will also only denote by Spc and Cat_∞ the ∞-categories of spaces and of ∞-categories. Given two objects X and Y of category $\mathscr{C}$ we will let $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ denote the mapping space between two objects, i.e. the object of the category Spc. When we need to use 1-categories we will explicitly see them as ∞-categories, via the nerve functor and we will say ordinary category or 1-category, to emphasize that their mapping spaces are equivalent to discrete sets.

Definition 2.3.13. Given a diagram $p : D \to \mathscr{C}$ from a category D[39] into a category $\mathscr{C}$ we say

[37] We haven't defined what these mean, see [33, Definition 5.5.6.8].

[38] We will refrain from discussing this technical condition here, all ∞-categories we will encounter in these notes will be presentable.

[39] From our conventions here D is an ∞-category, which can be represented by a simplicial set; one actually can define limits and colimits for any map from a simplicial set into a quasi-category $\mathscr{C}$ and this generality is sometimes useful in developing the theory.

- an object $X \in \mathscr{C}_{/p}$ is a *limit* of p if it is a terminal object of the category $\mathscr{C}_{/p}$;
- an object $X \in \mathscr{C}_{p/}$ is a *colimit* of p if it is an initial object of the category $\mathscr{C}_{/p}$.

Example 2.3.14. The category Cat_∞ admits limits and colimits. Say a bit more why this is true.

Example 2.3.15 (Spectra)**.** Let Spc_* denote the category of pointed spaces. We define the *loop (endo-)functor* on the category Spc_* by

$$\begin{aligned} \Omega : \mathrm{Spc}_* &\to \mathrm{Spc}_* \\ X &\mapsto * \underset{X}{\times} *. \end{aligned}$$

The *category of spectra* can be defined as the following limit in the category Cat_∞:

$$\mathrm{Spctr} := \left(\cdots \xrightarrow{\Omega} \mathrm{Spc}_* \xrightarrow{\Omega} \mathrm{Spc}_* \right).$$

This is also denoted by $\Omega(X) := X[-1]$.
Similarly, we define the suspension functor by

$$\begin{aligned} \Sigma : \mathrm{Spc} &\to \mathrm{Spc} \\ X &\mapsto * \underset{X}{\sqcup} *. \end{aligned}$$

Definition 2.3.16. Let $\imath : \mathscr{C}_0 \to \mathscr{C}$ be a functor[40] between ∞-categories and $\mathscr{D}$ another ∞-category . Then the functors of *left (resp. right) Kan extension* with respect to $\imath$ are the *partially* defined left (resp.) right adjoint to the restriction map

$$\mathrm{Fun}(\mathscr{C}, \mathscr{D}) \xrightarrow{(-)\circ \imath} \mathrm{Fun}(\mathscr{C}_0, \mathscr{D}).$$

The following is a good criterion to checking when $\mathrm{LKE}_\imath$ or $\mathrm{RKE}_\imath$ exist.

Lemma 2.3.17. *Let $F : \mathscr{C}_0 \to \mathscr{D}$ and $\imath : \mathscr{C}_0 \to \mathscr{C}$ be functors between ∞-categories. Then*

(a) if for all $X \in \mathscr{C}$ the colimit[41]

$$colim_{\mathscr{C}_0 \underset{\mathscr{C}}{\times} \mathscr{C}^{/X}} F \tag{2.7}$$

exists, then $LKE_\imath F$ exists. Moreover, in this case for every $X \in \mathscr{C}$ one has $LKE_\imath F(X)$ is computed by (2.7).

(b) if for all $X \in \mathscr{C}$ the limit

$$lim_{\mathscr{C}_0 \underset{\mathscr{C}}{\times} \mathscr{C}^{X/}} F \tag{2.8}$$

exists, then $RKE_\imath F$ exists. Moreover in this case for every $X \in \mathscr{C}$ one has $LKE_\imath F(X)$ is computed by (2.8).

Proof. This is [33, Proposition 4.3.2.15]. □

[40] Despite the notation we don't assume that $\imath$ is fully faithful.
[41] Here we simply write F for the composite

$$\mathscr{C}_0 \underset{\mathscr{C}}{\times} \mathscr{C}^{/X} \to \mathscr{C}_0 \xrightarrow{F} \mathscr{D}$$

where the first map is the canonical projection.

2.3.5 coCartesian Fibrations and Grothendieck construction

One of the most important concepts in developing the theory of higher categories is that of (co)Cartesian fibrations. In many situations in the theory one would like to write down a functor (of ∞-categories)

$$F : \mathscr{C} \to \mathscr{C}\mathrm{at}_\infty \tag{2.9}$$

from an arbitrary ∞-category $\mathscr{C}$ into the ∞-category of ∞-categories. This, however involves the specification of lots of *data*, since one needs to specific not only what this functor does to morphisms but also to where it sends all the higher coherence data.

Unsurprisingly this can be a hard task. The idea is encode the data of either of the functors in (2.9) in an ∞-category over $\mathscr{C}$, i.e. a morphism $p : \mathscr{D} \to \mathscr{C}$ from some ∞-category $\mathscr{D}$ to $\mathscr{C}$. The answer to what property the morphism (i.e. functor of ∞-categories) p has that corresponds to a functor into $\mathscr{C}\mathrm{at}_\infty$ (resp. $\mathscr{S}\mathrm{pc}$) is that of a coCartesian (resp. left[42]) fibration.

Before giving the definition, let's look at an example that might illustrate where some of the conditions in the definition come from. In fact, we will simplify the situation even further by considering a functor

$$F : \mathscr{C} \to \mathscr{S}\mathrm{pc}. \tag{2.10}$$

Let X be an object of an ∞-category $\mathscr{C}$. We know that given any object Y in $\mathscr{C}$ by Definition-Proposition 2.2.15 one has a space $F(X) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$, i.e. an object of $\mathscr{S}\mathrm{pc}$. We would like to formulate that this assignment is functorial in Y, for instance given a morphism $f : Z \to Y$ we want a map

$$\varphi : \mathrm{Hom}_{\mathscr{C}}(X, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Y).$$

We could naïvely pose

$$\varphi(g) := f \circ g. \tag{2.11}$$

However, there are two problems with writing a formula as (2.11):

(i) the composition $f \circ g$ is only well-defined as an object in the homotopy category $\mathrm{h}\mathscr{C}$;

(ii) suppose one is given another morphism $\alpha : Z' \to Z$ we want to be able to recover

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Z') &\to \mathrm{Hom}_{\mathscr{C}}(X, Y) \\ h &\mapsto (f \circ \alpha) \circ h \end{aligned}$$

from

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Z') &\to \mathrm{Hom}_{\mathscr{C}}(X, Y) \\ g &\mapsto f \circ g. \end{aligned}$$

However, this is not possible, since we are not keeping track of the higher data about that witness the composition $f \circ g$.

Let's for a second shift the perspective and suppose that we want to consider the data of $p : \mathscr{D} \to \mathscr{C}$, where $p^{-1}(Y) := \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Question: What property does the map of ∞-categories p have that encodes the functoriality that we want from the construction $Y \rightsquigarrow \mathrm{Hom}_{\mathscr{C}}(X, Y)$?

We want:

(i) for every morphism $f : Z \to Y$ in $\mathscr{C}$ and $g \in p^{-1}(Z) = \mathrm{Hom}_{\mathscr{C}}(X, Z)$ a morphism

$$\varphi : g \to f_*(g) \ \text{ in the category } \mathscr{D}$$

such that $p(f_*(g)) = Y$, i.e. $f_*(g)$ is an object of $p^{-1}(Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$. One can see that $f_*(g)$ is the object that plays the role of the ill-defined composition "$f \circ g$".

[42] In [16] the term coCartesian fibration in spaces is used for a left fibration.

(ii) Moreover, given a morphism $\alpha : D \to D'$ in $\mathscr{D}$ and E an object of $\mathscr{D}$ we want the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{D}}(D', E) & \xrightarrow{(-)\circ\alpha} & \mathrm{Hom}_{\mathscr{D}}(D, E) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Hom}_{\mathscr{C}}(p(D'), p(E)) & \xrightarrow{(-)\circ p(\alpha)} & \mathrm{Hom}_{\mathscr{D}}(p(D), p(E)). \end{array} \tag{2.12}$$

to be a pullback in the ∞-category $\mathscr{S}$pc.

We notice that condition (ii) above is circular since we were trying to define the maps $(-) \circ \alpha$ in the ∞-category $\mathscr{S}$pc to start with. It turns out that it is enough to require that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{D}}(D', E) & \xrightarrow{(-)\circ[\alpha]} & \mathrm{Hom}_{\mathscr{D}}(D, E) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Hom}_{\mathscr{C}}(p(D'), p(E)) & \xrightarrow[(-)\circ[p(\alpha)]]{} & \mathrm{Hom}_{\mathscr{D}}(p(D), p(E)). \end{array}$$

is a homotopy pullback diagram in the category h $\mathscr{S}$pc.

For concreteness we give a definition of left fibration in the model of quasi-categories.

Definition 2.3.18. A map $p : \mathscr{D} \to \mathscr{C}$ of ∞-categories (or even of simplicial sets) is a *left fibration* (or a *coCartesian fibration in spaces*) if for every $n \geq 1$ and $0 \leq i < n$ the dotted arrow making the following diagram commutative exists

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathscr{D} \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & \mathscr{C}. \end{array}$$

Remark 2.3.19. The notion of left fibration has an analogue in classical category theory, it corresponds to the notion of an opfibration in groupoids (see [34, Tag 015A] for a definition) and a functor $F : \mathsf{C} \to \mathsf{D}$ of ordinary categories is an opfibration in groupoids if and only if $\mathrm{N}_\bullet(F) : \mathrm{N}_\bullet \mathsf{C} \to \mathrm{N}_\nu \mathsf{D}$ is a left fibration of ∞-categories (see [34, Tag 015H]).

The main result proved in [33, §2.1] is the following:

Theorem 2.3.20. *Let $\mathscr{C}$ be an ∞-category. There exists an ∞-category $\mathscr{L}Fib_{/\mathscr{C}}$ of left fibrations over $\mathscr{C}$ and an equivalence of ∞-categories*

$$Gr : \mathscr{L}Fib_{/\mathscr{C}} \xrightarrow{\simeq} Fun(\mathscr{C}, \mathscr{S}pc).$$

Let's more formally address the question of the functoriality of the construction that sends two objects X and Y from an ∞-category $\mathscr{C}$ to the space of morphisms $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. In other words we want to encode a functor

$$\mathrm{Hom}_{\mathscr{C}}(-, -) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{S}\mathrm{pc}.$$

We will construct a left fibration $\mathrm{Tw}(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ where the $\mathrm{Tw}(\mathscr{C})$ is the so-called ∞-category of twisted arrows. Recall that given two linearly ordered sets I and J we define $I \star J := I \sqcup J$ as the set with the unique linear order that restrict to the orders in I and J and such that for every $i \in I$ and $j \in J$ one has $i \leq j$. Consider the functor

$$\begin{aligned} Q : \Delta &\to \Delta^{\mathrm{op}} \\ [n] &\mapsto [n] \star [n]^{\mathrm{op}} = [2n+1]. \end{aligned}$$

The ∞-*category of twisted arrows* $\mathrm{Tw}(\mathscr{C})$ is the simplicial set given by

$$\mathrm{Tw}(\mathscr{C})_n := \mathscr{C} \circ Q([n]).$$

Remark 2.3.21. Concretely, we have the following:

- $\mathrm{Tw}(\mathscr{C})_0$ is equivalent to $\mathscr{C}$;
- $\mathrm{Tw}(\mathscr{C})_1$ has as objects 3-simplices in $\mathscr{C}$, i.e. the following commutative diagrams:

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_1 \end{array}$$

- $\mathrm{Tw}(\mathscr{C})_2$ has as objects 5-simplices in $\mathscr{C}$, i.e. the following commutative diagrams:

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longleftarrow & Y_1 & \longleftarrow & Y_2 \end{array}$$

One has a natural map of simplicial sets:

$$\lambda : \mathrm{Tw}(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$$
$$(X \xrightarrow{f} Y) \mapsto (Y, X).$$

Proposition 2.3.22. *(a) The map λ is a left fibration. In particular one obtains that $Tw(\mathscr{C})$ is an ∞-category;*

(b) The equivalence of Theorem 2.3.20 determines a functor $Gr(\lambda) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Spc$ which corresponds via the equivalence $Fun(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}, \mathscr{S}pc) \simeq Fun(\mathscr{C}, Fun(\mathscr{C}^{\mathrm{op}}, \mathscr{S}pc))$ to the Yoneda embedding, i.e. the functor

$$h_{(-)} : \mathscr{C} \to Fun(\mathscr{C}^{\mathrm{op}}, \mathscr{S}pc)$$
$$X \mapsto Hom_{\mathscr{C}}(-, X).$$

Let's now return to our original problem that is to describe the data of a functor as in (2.9).

Definition 2.3.23. Let $p : \mathscr{D} \to \mathscr{C}$ be a morphism of ∞-categories a morphism $f : D \to D'$ in $\mathscr{D}$ is said to be *coCartesian* over $\mathscr{C}$ if for every object E in $\mathscr{D}$ the map

$$\mathrm{Hom}_{\mathscr{D}}(D', E) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}}(D, E) \underset{\mathrm{Hom}_{\mathscr{C}}(p(D), p(E))}{\times} \mathrm{Hom}_{\mathscr{C}}(p(D'), p(E))$$

is an equivalence. In this case we say that $f : D \to D'$ is a *(p-)coCartesian lift of* $p(f) : p(D) \to p(D')$.

Definition 2.3.24. A map $p : \mathscr{D} \to \mathscr{C}$ is a *coCartesian fibration* if for every morphism $f : C \to C'$ in $\mathscr{C}$ and D and object of $\mathscr{D}$ such that $p(D) \simeq C$, there exists a coCartesian lift of f, i.e. a coCartesian morphism $\tilde{f} : D \to D'$ in $\mathscr{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} p(D) & \xrightarrow{p(\tilde{f})} & p(D') \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ C & \xrightarrow{f} & C' \end{array}$$

This feels a bit different than [32]*2.1.1.10, i.e. no inert condition on the morphism in $\mathscr{C}$ and there is nothing analogous to condition (2) in *loc. cit.*

The main result about coCartesian fibrations is the following:

Theorem 2.3.25. *Let $\mathscr{C}$ be an ∞-category , there exists an equivalence of categories*

$$Gr : Fun(\mathscr{C}, \mathscr{C}at_\infty) \xrightarrow{\simeq} \mathscr{C}oFib(\mathscr{C}), \tag{2.13}$$

where $\mathscr{C}oFib(\mathscr{C})$ is the ∞-category of coCartesian fibrations over $\mathscr{C}$[43].

Remark 2.3.26. The natural diagram that one can write involving the equivalences of Theorem 2.3.20 and 2.3.25 commutes.

Remark 2.3.27. We emphasize that the advantage of constructing a coCartesian over an ∞-category $\mathscr{C}$ fibration instead of writing a functor from $\mathscr{C}$ into the ∞-category $\mathscr{C}\mathrm{at}_\infty$ is that one reduces the problem of specific all the *data* involved in a functor $\mathscr{C} \to \mathscr{C}\mathrm{at}_\infty$ into checking certain *conditions* on a map of ∞-categories $\mathscr{D} \to \mathscr{C}$.

We can informally describe the functor Gr in the equivalence (2.13) as well as a functor in the opposite direction.

Straightening: the functor $\mathbf{St} : \mathscr{C}\mathrm{oFib}(\mathscr{C}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{C}\mathrm{at}_\infty)$ can be described as follows. Let $p : \mathscr{D} \to \mathscr{C}$ be a coCartesian fibration the value of $\mathbf{St}(\mathscr{D})$ on objects is given by

$$\mathbf{St}(\mathscr{D})(C) := p^{-1}(X)$$

and on morphisms it is given by the coCartesian lifts of morphisms in $\mathscr{C}$.

Unstraightening: the functor $\mathbf{Un} : \mathrm{Fun}(\mathscr{C}, \mathscr{C}\mathrm{at}_\infty) \to \mathscr{C}\mathrm{oFib}(\mathscr{C})$ sends a functor $F : \mathscr{C} \to \mathscr{C}\mathrm{at}_\infty$ to the category whose:

- objects are pairs (C, D), where C is an object of $\mathscr{C}$ and D is a object of $F(C)$;
- a morphism between objects (C, D) and (C', D') are pairs $f \in \mathrm{Hom}_{\mathscr{C}}(C, C')$ and $g \in \mathrm{Hom}_{F(C')}(F(f)(D), D')$.

2.3.6 Adjoint Functor Theorem

2.4 Complements on Higher categories

In this section I will collect a number of constructions in higher categories that will be useful for during these Lectures. Some of them are not strictly necessary but will be helpful from a conceptual or computational point of view.

2.4.1 n-categories

Definition 2.4.1. Let $n \geq 0$, an ∞-category $\mathscr{C}$ is said to be an n-category if any of the following equivalent condition hold:

(1) for any two objects $X, Y \in \mathscr{C}$ the mapping space $\mathrm{Map}_{\mathscr{C}}(X, Y)$ is $(n-1)$-truncated;

(2) for every $m > n$ any diagram

$$\begin{array}{ccc} \Lambda^m_i & \longrightarrow & \mathscr{C} \\ \big\downarrow & \nearrow & \\ \Delta^m & & \end{array}$$

admits an unique dotted arrow making the diagram commutative;

(3) for any simplicial set K given two morphisms $f, f' : K \to \mathscr{C}$ then $f \simeq f'$ if and only if $f|_{\mathrm{sk}_n K} \simeq f'|_{\mathrm{sk}_n K}$, where $\mathrm{sk}_n K$ denotes the simplicial subset of K generated by the simplices of dimension $\leq n$.

(4) (...)

Can I formulate conditions (2) and (3) in a model independent way?

Give some arguments or references for the above.

[43] We will note precisely define this category, but we mention that the morphisms are maps of categories over $\mathscr{C}$ that send coCartesian morphisms to coCartesian morphisms.

2.4.2 Groupoid objects

Definition 2.4.2. Given an ∞-category $\mathscr{C}$ a *groupoid object* in $\mathscr{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathscr{C}$ satisfying any of the following equivalent conditions:

(1) for every $n \geq 0$ and every pair of simplicial sets S, S', such that $S \cup S' = [n]$ and $S \cap S' = \{s\}$ the following diagram

$$\begin{array}{ccc} X(n) & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ X(S') & \longrightarrow & X(s) \end{array}$$

is a pullback diagram in $\mathscr{C}$.

Notation 2.4.3. Let Δ_+ denote the ∞-category generated adjoining an initial object $+$ to Δ. An *augmented simplicial object* is a functor $X_+^\bullet : \Delta_+^{\mathrm{op}} \to \mathscr{C}$.

For any $n \geq 0$, we let $\Delta_+^{\leq n}$ denote the subcategory of Δ_+ generated by simplicies of dimension $\leq n$. For instance, $\Delta_+^{\leq 0} \simeq \{+ \to 0\}$.

Example 2.4.4. Let $\mathscr{C}$ be an ∞-category which admits finite limits. Given any morphism $f : X \to Y$ in $\mathscr{C}$ the *Čech nerve* of f is defined as the augmented simplicial object

$$(X/Y)_+^\bullet : \Delta_+^{\mathrm{op}} \to \mathscr{C}$$

obtained as the *right* Kan extension of the functor $\{0 \to +\} \simeq (\Delta_+^{\leq 0})^{\mathrm{op}} \to \mathscr{C}$ determined by $X \to Y$ via the canonical inclusion $(\Delta_+^{\leq 0})^{\mathrm{op}} \hookrightarrow \Delta_+^{\mathrm{op}}$. Informally[44], one has that

$$(X/Y)_+^n := X \underset{Y}{\times} X \underset{Y}{\times} \cdots \underset{Y}{\times} X$$

where one has $(n+1)$ copies of X.

We claim that $(X/Y)^\bullet := (X/Y)_+^\bullet|_{\Delta^{\mathrm{op}}}$ is a groupoid object.

Indeed, this follows from [33, Proposition 4.2.3.8]. Formulate the statement of this proposition below in a model-independent way.

Remark 2.4.5. Condition ((1)) in Definition 2.4.2 for small $n \geq 0$ can be explicitly spelled out as follows:

a) for $S = \{0 \to 1\}$ and $S' = \{1 \to 2\}$ one has

$$X([2]) \xrightarrow{\simeq} X([1]) \underset{X([0])}{\times} X([1])$$

where the morphisms are the restriction to the value of X on the vertex $\{1\}$;

b) for a decomposition of Δ^3 as $S = \{0 \to 1 \to 2\}$ and $S' = \{2 \to 3\}$ one has

$$X([3]) \xrightarrow{\simeq} X([2]) \underset{X([0])}{\times} X([1])$$

where the morphisms correspond to the restriction to the vertex $\{2\}$.

An important class of groupoid objects in an ∞-category is that of effective groupoid objects, informally these are the higher categorical analogue of effective equivalent relations[45]

Definition 2.4.6. A groupoid object $X^\bullet : \Delta^{\mathrm{op}} \to \mathscr{C}$ in an ∞-category $\mathscr{X}$ is said to be *effective* if there exists an augmented simplicial object $\tilde{X}^\bullet : \Delta_+^{\mathrm{op}} \to \mathscr{C}$ such that

[44] Notice that to define the Čech extension by these formulas we would be lacking the coherence data encoded in the higher morphisms, e.g. the different composites $(X/Y)_+^2 \to (X/Y)_+^0$, and also the witnesses of their relations, etc.

[45] Recall an equivalence relation $\mathsf{X}_1 \to \mathsf{X}_0 \times \mathsf{X}_0$ in an ordinary category C (assumed to have finite limits and coequalizers) is *effective* if the canonical morphism $\mathsf{X}_1 \to \mathsf{X}_0 \underset{\mathsf{X}_{-1}}{\times} \mathsf{X}_0$, where $\mathsf{X}_{-1} := \mathrm{Coeq}(\mathsf{X}_1 \rightrightarrows \mathsf{X}_0)$.

a) it extends $X^\bullet$, i.e. $\tilde{X}^\bullet_+\big|_{\Delta^{\mathrm{op}}} \simeq X^\bullet$;

b) $\tilde{X}^\bullet$ is obtained as a Čech nerve (cf. Example 2.4.4), i.e. the canonical map

$$\tilde{X}^\bullet \to \mathrm{RKE}_{(\Delta_+^{\leq 0})^{\mathrm{op}} \hookrightarrow \Delta_+^{\mathrm{op}}}(\tilde{X}^\bullet\big|_{(\Delta_+^{\leq 0})^{\mathrm{op}}})$$

is an equivalence.

2.4.3 Tools for computing limits and colimits

One of the points that probably scares people in doing arguments with ∞-categories is the absence or inadequacy of formulas and arguments with elements. In this section we try to collect, in a model-independent formulation, some results that allow one to simplify computations of limits and colimits.

Decomposition of diagrams

Let $\mathscr{K}$ denote an ∞-category and $p : \mathscr{K} \to \mathscr{C}$ a functor whose limit or colimit we want to study, starting with the question of existence.

Let J be an ordinary category, which we see as ∞-category in a natural way, and consider a functor $F : \mathsf{J} \to (\mathrm{Cat}_\infty)_{/\mathscr{K}}$.

I don't quite know what would be the model-independent formulation of [33, Proposition 4.2.3.4]. Come back to this!

Notation 2.4.7. Given $S \to \mathsf{K}$ a morphism from a space S to K let[46]

$$\mathsf{J}_S := (\mathsf{J} \underset{\mathsf{K}}{\times} S)^{\simeq}.$$

Understand the model-independent meaning of $\mathscr{J}'_K$ in [33, Notation 4.3.2.7].

[46] Notice that the functor $S \to \mathsf{K}$ naturally factors through the underlying groupoid $\mathsf{K}^{\simeq} \to \mathsf{K}$, however this does not guarantee that the fiber product $\mathsf{J} \underset{\mathsf{K}^{\simeq}}{\times} S$ is a groupoid, since K is not a groupoid.

Technical aspects of Chapter 1

In this part we collect technical results regarding the material of ∞-categories.

2.5 Models of ∞-categories

2.5.1 Model categories

2.5.2 ∞-category underlying a model category

There are two cases of this construction:

- assume that M is a simplicial model category, i.e. see [43, §II.2] or [33, Definition A.3.1.5.], then the *underlying* ∞*-category of* M is $N^{\mathrm{hc}}_\bullet(\mathsf{M}_{\mathrm{cf}})$, where M_{cf} is the subcategory of fibrant and cofibrant objects, which satisfies the condition of Proposition 2.2.41 because of Quillen's axiom (SM7);
- assume that M is an *arbitrary* model category, then by forgetting the fibrations and cofibrations one obtains a relative category (M, W) where W is the class of weak equivalences, the *underlying* ∞*-category of* M is the ∞-category associated to this relative category (see Remark 2.2.63 below).

There is a further important special case.

2.5.3 Equivalent models of ∞-categories

Proposition 2.5.1. *In the category of simplicial categories with the model structure described above (or in [7, Theorem 4.3.2]) a simplicial category $\mathscr{C}$ is fibrant if and only if for every $X, Y \in \mathscr{C}$ the simplicial set*

$$Maps_{\mathscr{C}}(X, Y)$$

is a Kan complex, i.e. fibrant in the Kan model structure on simplicial sets.

2.5.4 Comparision of constructions of Spc

The most important ∞-category one can consider is the ∞-category of spaces Spc. There are many possible constructions of it, essentially by specializing the defining idea of each model to the case of topological spaces. In this subsection we compare some of these definitions, which can be seen as a toy model in comparing each of the models.

A straightforward definition of Spc in the model of quasi-categories is as

$$\mathrm{Spc} := \mathrm{NKan},$$

where $\mathrm{Kan} \subset \mathrm{Set}_\Delta$ is the subcategory of Kan complexes.

In [19, Appendix A] there is a nice discussion of quotient ∞-categories. We explain it here. Let $\mathsf{Top}^{\mathrm{CW}}$ denote the 1-category of CW-complexes, seen as an ∞-category, i.e. by taking its nerve. One can consider the following cosimplicial object

$$\Delta_{\mathrm{Top},\bullet} : \Delta \to \mathsf{Top}^{\mathrm{CW}}$$
$$[n] \mapsto \Delta^n_{\mathrm{top}}$$

where $\Delta^n_{\text{top}} := \{(x_i)_{0\le i\le n} \mid \sum_{i=0}^n x_i = 1\}$ is the topological n-simplex. This gives a simplicial object $\mathsf{Top}^{\text{CW}}[\Delta^\bullet_{\text{Top}}]$ in ∞-categories, where $\mathsf{Top}^{\text{CW}}[\Delta^n_{\text{Top}}]$ is defined as the 1-category whose objects are the same as Top^{CW} but morphisms as given by

$$\text{Hom}_{\mathsf{Top}^{\text{CW}}[\Delta^n_{\text{Top}}]}(X,Y) := \text{Hom}_{\mathsf{Top}^{\text{CW}}}(\Delta^n_{\text{top}} \times X, Y).$$

That this is well-defined follows from the fact that Top^{CW} is a Cartesian closed symmetric monoidal category[47]. Let

$$\text{Spc}' := \left|\mathsf{Top}^{\text{CW}}[\Delta^\bullet_{\text{Top}}]\right|$$

denote the geometric realization in the ∞-category of ∞-categories of the above simplicial object.

Lemma 2.5.2. *One has an equivalence of ∞-categories*

$$Spc' \simeq Spc.$$

Proof. Give details about this! □

2.6 Analytical higher category theory

In this section we present the some of the constructions of category theory notions for ∞-categories in the model of quasi-categories.

2.6.1 Limits, colimits and Kan extensions

2.6.2 CoCartesian fibrations and Grothendieck construction

2.6.3 Adjoint functor theorem

2.7 Synthetic higher category theory

In this section we present some of the ideas from [46] where the category theory of ∞-categories is developed precisely in a model-independent way.

2.7.1 Recollections on 2-category theory

Definition 2.7.1. Let $\mathbb{C}$ and $\mathbb{D}$ be two 2-categories a *lax functor* $F : \mathbb{C} \to \mathbb{D}$ is the data

- an object $F(X)$ of $\mathbb{D}$ for every X an object of $\mathbb{C}$;
- for every two objects X and Y in $\mathbb{C}$ a functor

$$F : \text{Hom}_{\mathbb{C}}(X,Y) \to \text{Hom}_{\mathbb{D}}(F(X), F(Y));$$

- a 2-morphism $\epsilon : F(\text{id}_X) \Rightarrow \text{id}_{F(X)}$;
- for every pair f, g of composable morphisms in $\mathbb{C}$ a composition constraint:

$$\mu_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f).$$

This data is required to satisfy the expected compatibility constraints, i.e. the different ways to apply the composition constraint to a triple of composable morphism should agree and the 2-morphisms relating the image of the identity morphism to the identity morphism of the image should be compatible with the composition constraint.

[47] We can realize $\mathsf{Top}^{\text{CW}}[\Delta^n_{\text{Top}}]$ as the subcategory of $\mathsf{Top}^{\text{CW}}_{/\Delta^n_{\text{top}}}$ whose objects are isomorphic to $\Delta^n_{\text{top}} \times X$ for some $X \in \mathsf{Top}^{\text{CW}}$.

Definition 2.7.2. A lax functor $F : \mathbb{C} \to \mathbb{D}$ is said to be *unitary* if for all objects X the 2-morphism ϵ_X is an isomorphism. Moreover, we say that F is *strictly unitary* if for all X an object of $\mathbb{C}$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and the identity constraint ϵ_X is the identity 2-morphism of $\mathrm{id}_{F(X)}$.

Construction 2.7.3. Let $[n]$ denote the partially ordered set $\{0 < 1 < \cdots < n\}$, which we regard as a 2-category with only identity 2-morphisms. Given a 2-category $\mathbb{C}$ we define

$$\mathrm{N}^{\mathrm{D}}_n \mathbb{C} := \{ \text{ strictly unitary lax functors } F : [n] \to \mathbb{C} \text{ } \}.$$

This assignment is clearly functorial on $[n]$, thus we obtain a simplicial set $\mathrm{N}^{\mathrm{D}}_\nu(\mathbb{C})$.

2.7.2 ∞-cosmoi

2.7.3 Limits, colimits and Kan extensions

Chapter 3

Stable categories and ∞-operads

Idea: This chapter should be renamed complements on ∞-categories. It should include stable categories, ∞-operads and presentable categories: really results involving Ind-construction, compact and projective objects.

In this chapter we give a brief discussion of two notions:

1. that of an stable ∞-category;
2. the notion of ∞-operads.

The first is the natural generalization to ∞-categories of the notion of an abelian category. It is very useful in the same sense that abelian categories are useful: they have finite produts and coproducts and these coincide.

The second notion is important to make sense of the notion of a symmetric structure in the context of ∞-categories. Normally, in the study of commutative rings and their modules, one defines these objects and the category of commutative rings and R-modules for an ordinary commutative ring R and one checks that one can define tensor product operations and finally one phrases the whole structure as a symmetric monoidal category. The notion of operads (or colored operads) then shows up only when one is interested in more general type of algebras than commutative and associative. I

In the context of ∞-categories one pursues these construction in a reverse order. Firstly, due to the fact that defining a commutative algebra up to homotopy involves lots of coherent data–one needs to formalize the types of ∞-categories where one should make sense of that: this leads to the notion of ∞-operads[1]. Incidentally, the gadgets that keeps track of the homotopy coherent be it associative or commutative algebra or that keep track of the data of an action of an algebra on a module all form ∞-operads. Secondly, one considers a special type of ∞-operads which keep track of an ∞-category with a (symmetric) monoidal structure[2]. Lastly, one uses maps between ∞-operads to formulate the notions of commutative or associative algebra objects in an ∞-category and also their modules in an ∞-category.

3.1 ∞-operads

Instead of considering the whole theory of ∞-operads we will restrict ourselves to a discuss of monoidal, symmetric monoidal and module categories.

3.1.1 Symmetric monoidal ∞-categories

The definition of a symmetric monoidal structure on an ∞-category is an application of Theorem 2.3.25. Before we explain the ∞-categorical case let's revisit the notion of symmetric monoidal ordinary category from a (possibly) different viewpoint than must people think about it.

[1]Notice that in usual ring theory we use the symmetric monoidal structure of sets to write down the multiplication map of a commutative ring.

[2]Actually here the theory allows for novel notions, namely that of $\mathbb{E}_n$-categories–these interpolate between only a monoidal structure and the data of a commutative constraint.

Let Fin_* denote the ordinary category of finite pointed sets. For each integer $n \geq 0$ we denote by $\langle n\rangle$ the set $\{0, 1, \ldots, n\}$ pointed by 0. We recall that if C is an ordinary category, the data of a symmetric monoidal structure on C, i.e. a functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, an unit object $\mathbb{1}_{\mathsf{C}}$ in C, and the associativity and commutativity constraints is equivalent to a functor

$$\mathsf{C}^{\otimes} : \mathrm{Fin}_* \to \mathrm{Cat} \tag{3.1}$$

into the 1-category of categories Cat that satisfy the following conditions:

- $\mathscr{C}^{\otimes}(\langle 0\rangle) \simeq *$;
- the maps $e_i : \langle n\rangle \to \langle 1\rangle$ defined by $e_i(j) = 1$ if $i = j$ and 0 if $i \neq j$ assemble to give a morphism $\langle n\rangle \to \langle 1\rangle^{\times n}$, we require that for all $n \geq 1$ the associated morphism in Cat:

  $$\mathscr{C}^{\otimes}(\langle n\rangle) \xrightarrow{\simeq} \prod_{i=1}^{n} \mathscr{C}^{\otimes}(\langle 1\rangle)$$

  is an isomorphism, i.e. an equivalence of categories.

By the classical version of Theorem 2.3.25 the data of (3.1) determines a coCartesian fibration of ordinary categories

$$\mathsf{C}^{\otimes, \mathrm{Fin}_*} \to \mathrm{Fin}_*.$$

Remark 3.1.1. (Insert reference about coCartesian fibrations in usual category theory.)

Definition 3.1.2. Let $\mathscr{C}$ be an ∞-category, the data of a symmetric monoidal structure on $\mathscr{C}$ is that of a coCartesian fibration

$$p : \mathscr{C}^{\otimes, \mathrm{Fin}_*} \to \mathrm{Fin}_*$$

satisfying the conditions:

- $\mathscr{C}^{\otimes, \mathrm{Fin}_*}_{\langle 1\rangle} := p^{-1}(\langle 1\rangle) \simeq \mathscr{C}$;
- $\mathscr{C}^{\otimes, \mathrm{Fin}_*}_{\langle 0\rangle} \simeq *$;
- the canonical morphisms $e_i : \langle n\rangle \to \langle 1\rangle^{\times n}$ determines an equivalence of ∞-categories:

  $$\mathscr{C}^{\otimes, \mathrm{Fin}_*}_{\langle n\rangle} \xrightarrow{\simeq} \prod_{i=1}^{n} \mathscr{C}^{\otimes, \mathrm{Fin}_*}_{\langle 1\rangle}.$$

Notice that the functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ can be obtained as follow. Consider the functor $\mathscr{C}^{\otimes} : Finp \to \mathrm{Cat}_\infty$ associated to the coCartesian fibration $\mathscr{C}^{\otimes, \mathrm{Fin}_*} \to \mathrm{Fin}_*$ one has the morphisms

$$\langle 1\rangle \times \langle 1\rangle \xleftarrow{e_1 \times e_2} \langle 2\rangle \xrightarrow{\mu} \langle 1\rangle\,,$$

where $e_1 \times e_2$ was explained above and $\mu : \langle 2\rangle \to \langle 1\rangle$ is defined by $\mu(1) = \mu(2) = 1$. Applying the functor $\mathscr{C}^{\otimes}$ we obtain

$$\mathscr{C} \times \mathscr{C} \simeq \mathscr{C}^{\otimes}(\langle 1\rangle) \times \mathscr{C}^{\otimes}(\langle 1\rangle) \xleftarrow{\simeq} \mathscr{C}^{\otimes}(\langle 2\rangle) \to \mathscr{C}^{\otimes}(\langle 1\rangle) \simeq \mathscr{C}$$

and $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is defined as the composite above[3].

Example 3.1.3. Let $\mathscr{C}$ be an ∞-category that admits all finite products, in particular it has a final object, i.e. the empty product. Consider the functor

$$\begin{aligned} \Pi : \mathrm{Fin}_* &\to \mathrm{Cat}_\infty \\ \langle n\rangle &\mapsto \mathrm{Fun}(\langle n\rangle, \mathscr{C}) \underset{\mathrm{Fun}(\langle 0\rangle, \mathscr{C})}{\times} \mathrm{pt}, \end{aligned}$$

[3]Notice the similarity with the idea that allows us to define the composition of morphisms in an ∞-category in 2.2.20.

where $\mathrm{pt} \to \mathrm{Fun}(\langle 0\rangle, \mathscr{C}) \simeq \mathscr{C}$ maps to the final object. This classifies a Cartesian fibration

$$\mathrm{Un}(\Pi) \to \mathrm{Fin}_*.$$

We claim that $\mathrm{Un}(\Pi)$ is also a coCartesian fibration[4]. Thus, let

$$\mathrm{St}^{\mathrm{coCart}}(\mathrm{Un}(\Pi)) : \mathrm{Fin}_* \to \mathrm{Cat}_\infty$$

denote the functor classified by $\mathrm{Un}(\Pi)$ seem as a coCartesian fibration. It is easy to check that $\mathscr{C}^{\otimes} := \mathrm{St}^{\mathrm{coCart}}(\mathrm{Un}(\Pi))$ satisfies the conditions of Definition 3.1.2[5].

Remark 3.1.4. Any symmetric monoidal structure obtained as in Example 3.1.3 is called a *Cartesian symmetric monoidal structure*, since the coCartesian fibration defining the symmetric monoidal structure is also a Cartesian fibration. In particular, since Cat_∞ and Spc admit finite limits one has Cartesian symmetric monoidal structures on these ∞-categories.

To formulate the notion of strict and lax symmetric monoidal functors we introduce the following useful concept.

Definition 3.1.5. A morphism $f : \langle n\rangle \to \langle m\rangle$ in Fin_* is said to be *idle* if its restriction $f|_{\langle n\rangle\setminus f^{-1}(0)}$ is injective, i.e. for every element $j \in \langle m\rangle \setminus\{0\}$ the inverse image $f^{-1}(j)$ has at most one element. Moreover, given a coCartesian fibration $p : \mathscr{E} \to \mathrm{Fin}_*$ a coCartesian morphism in $\mathscr{C}$ is said to be *idle* if its image under p is idle.

Definition 3.1.6. A map of coCartesian fibrations $F : \mathscr{C}^{\otimes,\mathrm{Fin}_*} \to \mathscr{D}^{\otimes,\mathrm{Fin}_*}$ is said to be:

- a *symmetric monoidal functor* if it sends all coCartesian morphisms to coCartesian morphisms;
- a *right-lax symmetric monoidal functor* if it sends idle coCartesian morphisms to coCartesian morphisms;
- a *left-lax symmetric monoidal functor* if the corresponding map $F^{\mathrm{Cart}} : \mathscr{C}^{\otimes,\mathrm{Fin}_*^{\mathrm{op}}} \to \mathscr{D}^{\otimes,\mathrm{Fin}_*^{\mathrm{op}}}$ between the corresponding Cartesian fibrations sends idle coCartesian morphisms to coCartesian morphisms.

Lemma 3.1.7. *Let $\mathscr{C}^{\otimes}$ and $\mathscr{D}^{\otimes}$ be two symmetric monoidal categories and*

$$F : \mathscr{C} \longleftrightarrow \mathscr{D} : G$$

be a pair of adjoint functors between the underlying categories. Then the data of a left-lax symmetric monoidal structure on F is equivalent to the data of a right-lax symmetric monoidal structure on G.

Proof. Write this! Exercise for now! □

One can now define the data of a commutative algebra object in a symmetric monoidal category.

Definition 3.1.8. Let $\mathscr{C}^{\otimes}$ be a symmetric monoidal ∞-category, the data of an *unital commutative algebra* object in $\mathscr{C}$ is that of a right-lax symmetric monoidal functor:

$$A : \mathrm{pt}^{\otimes,\mathrm{Fin}_*} \to \mathscr{C}^{\otimes,\mathrm{Fin}_*},$$

where the coCartesian fibration $\mathrm{pt}^{\otimes,\mathrm{Fin}_*}$ classifies the functor $\mathrm{pt}^{\otimes} : \mathrm{Fin}_* \to \mathrm{Cat}_\infty$ that sends all pointed sets $\langle n\rangle$ to the final category pt with a single object.

[4] Indeed, unwinding the definition we notice that it is enough to consider the morphisms $e_i : \langle 1\rangle \to \langle n\rangle$ and $\alpha_n : \langle n\rangle \to \langle 1\rangle$. For $Y \in \Pi(\langle 1\rangle)$ the data of a coCartesian lift of e_i is that of a set $\{X_1, \dots, X_n\}$ of n objects in $\mathscr{C}$ together with a morphism $Y \to X_i$, this is achieved by simply taking $X_i = Y$ and the identity morphism. Whereas for $\{X_1, \dots, X_n\} \in \Pi(\langle n\rangle)$ the data of a coCartesian lift of α_n is the data of an object $Y \in \mathscr{C}$ and morphisms $p_i : Y \to X_i$ for each $i \in \{1, \dots, n\}$; this is exactly achieve by taking Y to be the product of all the objects $\{X_1, \dots, X_n\}$.

[5] I.e. when they are translated in terms of the functor $\mathscr{C}^{\otimes} : \mathrm{Fin}_* \to \mathrm{Cat}_\infty$ (see Definition 3.1.14 below)

Remark 3.1.9. Let's unpack Definition 3.1.8 for small n.

Notice one has two inert morphisms $p_i : \langle 2\rangle \to \langle 1\rangle$ defined by $p_1(1) = 1$ and $p_2(2) = 1$, with all the other objects mapping to 0. The requirement that $A(p_i) : A(\langle 2\rangle) \to A(\langle 1\rangle)$ is coCartesian says that for any object $Z \in \mathscr{C}^{\otimes,\mathrm{Fin}_*}$ one has an equivalence

$$\mathrm{Hom}_{\mathscr{C}^{\otimes,\mathrm{Fin}_*}}(A(\langle 1\rangle), Z) \to \mathrm{Hom}_{\mathscr{C}^{\otimes,\mathrm{Fin}_*}}(A(\langle 2\rangle), Z) \underset{\mathrm{Hom}_{\mathrm{Fin}_*}(\langle 1\rangle, p(Z))}{\times} \mathrm{Hom}_{\mathrm{Fin}_*}(\langle 2\rangle, p(Z)).$$

Recall (see **??**) that since $\mathscr{C}^{\otimes,\mathrm{Fin}_*} = \mathrm{Un}(\mathscr{C}^{\otimes})$ one has the following description of objects of $\mathrm{Hom}_{\mathscr{C}^{\otimes,\mathrm{Fin}_*}}(A(\langle 1\rangle), Z)$ this is a pair:

$$f : \langle 1\rangle \to p(Z) \quad \text{and} \quad \tilde{f} : \mathscr{C}^{\otimes}(f)(A(\langle 1\rangle)) \to Z$$

where $\tilde{f}$ is a morphism in $\mathscr{C}^{\otimes}(p(Z))$. Similarly, a morphism in $\mathrm{Hom}_{\mathscr{C}^{\otimes,\mathrm{Fin}_*}}(A(\langle 2\rangle), Z)$ is the data of

$$g : \langle 1\rangle \to p(Z) \quad \text{and} \quad \tilde{g} : \mathscr{C}^{\otimes}(g)(A(\langle 2\rangle)) \to Z.$$

The condition that $(g, \tilde{g})$ belong to the right-hand side of (**??**) means that $g = f \circ e_1$ and that

$$\tilde{g} \simeq \tilde{f} \circ \mathscr{C}^{\otimes}(f)(e_1).$$

By considering all $Z \in \mathscr{C}^{\otimes}(\langle 1\rangle)$ we see that the objects

$$\mathscr{C}^{\otimes}(e_1)(A\langle 2\rangle) \simeq A\langle 1\rangle$$

are isomorphic in $\mathscr{C}^{\otimes}(\langle 1\rangle)$. Similarly, one gets that $\mathscr{C}^{\otimes}(e_2)(A\langle 2\rangle) \simeq A\langle 1\rangle$. By the description of $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ from Remark **??** we see obtain that

$$A\langle 2\rangle \simeq A\langle 1\rangle \otimes A\langle 1\rangle.$$

Now, notice that if we imposed that the map $\alpha : \langle 2\rangle \to \langle 1\rangle$ which sends 1 and 2 to 1 were sent to a coCartesian morphism, a similarly reasoning as above would give that

$$A\langle 2\rangle \simeq A\langle 1\rangle,$$

which we don't expect to hold in general.

Remark 3.1.10. Maybe a more natural definition of a commutative algebra object in $\mathscr{C}$

Example 3.1.11 (Sanity Check)**.** Let C be an ordinary symmetric monoidal category. Then one can define a coCartesian fibration $\mathsf{C}^{\otimes,\mathrm{Fin}_*} \to \mathrm{Fin}_*$ as follows:

(i) objects of $\mathsf{C}^{\otimes,\mathrm{Fin}_*}$ are pairs $(\langle n\rangle, \{X_i\}_{1\leq i\leq n})$, which map to $\langle n\rangle$ in Fin_*;

(ii) morphisms between $(\langle n\rangle, \{X_i\}_{1\leq i\leq n})$ and $(\langle m\rangle, \{Y_j\}_{1\leq j\leq m})$ is the data of a map $f : \langle n\rangle \to \langle m\rangle$ in Fin_* plus a morphism

$$\alpha : \bigotimes_{1\leq j\leq m} \left(\otimes_{i\in f^{-1}(j)} \overline{X}_i\right) \to \otimes_{1\leq j\leq m} Y_j,$$

where $\overline{X}_i := X_i$ if $f(i) \neq 0$ and $\overline{X}_i := \mathbb{1}_{\mathsf{C}}$ if $f(i) = 0$.

I will leave as an exercise to check that this gives a coCartesian fibration satisfying the conditions of Definition 3.1.2.

Though Example 3.1.11 shows that the theory of symmetric monoidal ∞-categories subsumes that of ordinary symmetric monoidal categories. Sometimes one might be given a symmetric monoidal functor in a more strict form but really encoding the tensor product between the underlying objects only up to homotopy. The paradigm example is that of a monoidal model structure. Before we state this example we need to mention a certain result about localization of ∞-categories.

Definition-Proposition 3.1.12. Let $\mathscr{C}$ be an ∞-category, a *system* on $\mathscr{C}$ is a collection of morphisms $W \subset \mathrm{Fun}([1], \mathscr{C})$ such that

- W is stable under homotopies;
- W is stable under compositions;
- W contains all equivalences.

In this case there exists an ∞-category $\mathscr{C}[W^{-1}]$ such that any map from $\mathscr{C}$ to an ∞-category $\mathscr{D}$ which sends all the elements of W to isomorphisms factors through $\mathscr{C}[W^{-1}]$.

Moreover, if $\mathscr{C}$ is a symmetric monoidal ∞-category and the collection of morphisms W is stable under taking tensor products on the left and on the right, then $\mathscr{C}[W^{-1}]$ can be endowed with the structure of a symmetric monoidal ∞-category and has the same universal property as above with respect to symmetric monoidal functors.

Proof. This is a somewhat formal result, once one makes sense of the procedure in Remark 2.2.63. Indeed, given a pair $(\mathscr{C}, W)$ of an ∞-category and a system W one has a marked simplicial set $(S_{\mathscr{C}}, W)$ representing it. By picking any fibrant replacement $(S_{\mathscr{C}}, W) \to (\mathscr{C}', W')$ in the (combinatorial) simplicial model structure of market simplicial sets, one defines:

$$\mathscr{C}[W^{-1}] := \mathscr{C}'.$$

The statement about symmetric monoidal structure is [32, Proposition 4.1.7.4]. □

Example 3.1.13. Let C be a symmetric monoidal model structure[6] then the ∞-category $N(\mathsf{C}_c)[W^{-1}]$[7] inherits the structure of a symmetric monoidal ∞-category. Of course, this ∞-category is rather tricky to describe explicitly, since we performed a fibrant replacement when using Definition-Proposition 3.1.12. The main goal of §[32, §4.1.7] is to describe this ∞-category as some form of nerve construction.

3.1.2 Monoidal ∞-categories

Let Δ^{op} denote the simplicial category seen as an ∞-category. For any $n \geq 1$ one has an unique morphism

$$\mathrm{sp}_n : [n] \to [1] \times \cdots \times [1] \tag{3.2}$$

given on the ith factor by $\rho_i : [n] \to [1]$ given by

$$\rho_i(j) = \begin{cases} 1 & \text{if } j = i; \\ 0 & \text{else.} \end{cases}$$

We will call $\mathrm{sp}_n : [n] \to [1]^{\times n}$ the *spine morphism*. By convention we will pose $\mathrm{sp}_0 = \mathrm{id}_{[0]}$ to be the identity morphism of $[0]$.

Definition 3.1.14. A *monoidal structure on an ∞-category $\mathscr{C}$* is the data of a functor

$$\mathscr{C}^{\otimes} : \Delta^{\mathrm{op}} \to \mathrm{Cat}_\infty \tag{3.3}$$

such that

- $\mathscr{C}^{\otimes}([0]) \simeq \mathrm{pt}$;
- for every $n \geq 1$ the morphism $\mathscr{C}^{\otimes}(\mathrm{sp}_n)$ is an isomorphism;
- $\mathscr{C}^{\otimes}([1]) \simeq \mathscr{C}$.

[6] I.e. 1_{C} is cofibrant, the monoidal structure on C is closed and $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a left Quillen bifunctor.

[7] Here W is the class of weak equivalences and $\mathsf{C}_c \subseteq \mathsf{C}$ denotes the subcategory of (only) cofibrant objects. This is where the monoidal structure behaves well with respect to the weak equivalences.

As usual we will just say that $\mathscr{C}$ is a monoidal ∞-category for all the data of the functor (3.3). We will write

$$\mathscr{C}^{\otimes,\Delta^{\mathrm{op}}} \to \Delta^{\mathrm{op}}$$

for the coCartesian fibration associated to (3.3).

Remark 3.1.15. A word for the perplexed reader of the different looking Definition 4.1.1.10 in [32] for a monoidal ∞-category $\mathscr{C}$. In *loc. cit.* a monoidal ∞-category is the data of a coCartesian fibration $p : \mathscr{C}^{\otimes'} \to \mathrm{Assoc}^{\otimes}$[8] by straightening the data of p one obtains a functor

$$\mathrm{St}(p) : \mathrm{Assoc}^{\otimes} \to \mathrm{Cat}_\infty$$

which satisfies the condition of Definition 2.4.2.1 [32], i.e. $\mathrm{St}(p)$ defines an Assoc-monoid object in Cat_∞. However, since the category Cat_∞ has finite products by [32, Proposition 4.1.2.10] the data of $\mathrm{St}(p)$ is equivalent to that of (3.3). The point is that both ∞-categories Δ^{op} and $\mathrm{Assoc}^{\otimes}$ give equivalent models for the correct notion of the associative ∞-operads, thought they are not equivalent as ∞-categories.

The description of monoidal categories using the associated coCartesian or Cartesian fibrations is useful to define the appropriate notion of monoidal functors.

Definition 3.1.16. Let $\mathscr{C}^{\otimes,\Delta^{\mathrm{op}}} \to \Delta^{\mathrm{op}}$ and $\mathscr{D}^{\otimes,\Delta^{\mathrm{op}}} \to \Delta^{\mathrm{op}}$ denote two monoidal categories:

- a *(strict) monoidal functor* is the data of a map $\alpha : \mathscr{C}^{\otimes,\Delta^{\mathrm{op}}} \to \mathscr{D}^{\otimes,\Delta^{\mathrm{op}}}$ over Δ^{op} which takes *any* coCartesian morphism in $\mathscr{C}^{\otimes,\Delta^{\mathrm{op}}}$ to a coCartesian morphism in $\mathscr{D}^{\otimes,\Delta^{\mathrm{op}}}$;
- a *right-lax monoidal functor* is the data of a map $\alpha : \mathscr{C}^{\otimes,\Delta^{\mathrm{op}}} \to \mathscr{D}^{\otimes,\Delta^{\mathrm{op}}}$ over Δ^{op} which takes coCartesian morphisms in $\mathscr{C}^{\otimes,\Delta^{\mathrm{op}}}$ over a spine morphism (3.2) to coCartesian morphisms in $\mathscr{D}^{\otimes,\Delta^{\mathrm{op}}}$;
- a *left-lax monoidal functor* is the data of a map $\alpha : \mathscr{C}^{\otimes,\Delta} \to \mathscr{D}^{\otimes,\Delta}$[9] over Δ which takes coCartesian morphisms in $\mathscr{C}^{\otimes,\Delta}$ over a spine morphism (3.2) to coCartesian morphisms in $\mathscr{D}^{\otimes,\Delta}$.

Remark 3.1.17. For a 1-category C seen as an ∞-category the notion of a monoidal structure from Definition 3.1.14 agrees with the usual notion of a monoidal structure in a 1-category. Similarly, for the notion of a monoidal functor between 1-categories C and D (see [18]*Lemma 2.2.12 for details).

The notion of a right-lax monoidal functor allows us to easily make sense of associative algebra objects in a monoidal category.

Definition 3.1.18. Let $\mathscr{C}^{\otimes,\Delta^{\mathrm{op}}}$ be a monoidal ∞-category. An *associative algebra object in* $\mathscr{C}$ is the data of a right-lax monoidal functor

$$\mathrm{pt}^{\otimes,\Delta^{\mathrm{op}}} \to \mathscr{C}^{\otimes,\Delta^{\mathrm{op}}},$$

where $\mathrm{pt}^{\otimes,\Delta^{\mathrm{op}}}$ is the terminal ∞-category with its trivial monoidal structure, i.e. $\mathrm{pt}^{\otimes}([n]) = \mathrm{pt}$[10].

3.2 Stable ∞-categories

3.2.1 Stable ∞-categories

We say that an ∞-category $\mathscr{C}$ is *pointed* if it admits an object $0 \in \mathscr{C}$ which is both initial and final. We call 0 the *zero object*. A diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array} \tag{3.4}$$

is called a:

[8] Here $\mathrm{Assoc}^{\otimes}$ is the associative ∞-operad, see [32, Definition 4.1.1.3] for the definition.
[9] Here $\mathscr{C}^{\otimes,\Delta}$ and $\mathscr{D}^{\otimes,\Delta}$ denote the Cartesian fibrations over Δ corresponding to the functors $\mathscr{C}^{\otimes}$ and $\mathscr{D}^{\otimes}$, respectively.
[10] Here $\mathrm{pt}^{\otimes} := \mathrm{St}(\mathrm{pt}^{\otimes,\Delta^{\mathrm{op}}})$.

- *fiber sequence* if it is a pullback diagram;
- *cofiber sequence* if it is a pushout diagram.

Given a morphism $f : X \to Y$ in $\mathscr{C}$ a *cofiber of* f is an object Z and a pushout diagram (3.4), similarly given a morphism $g : Y \to Z$ a *fiber of* g is an object X and a diagram as (3.4).

Definition 3.2.1. Let $\mathscr{C}$ be an ∞-category then $\mathscr{C}$ is a *stable* ∞-category if

(a) $\mathscr{C}$ is pointed;

(b) every morphism f has a fiber and cofiber;

(c) every fiber sequence is *also* a cofiber sequence.

Here is an important property of stable ∞-categories:

Proposition 3.2.2. *Given a pointed category $\mathscr{C}$ then $\mathscr{C}$ is stable if and only if the following conditions are satisfied*

(i) $\mathscr{C}$ admits finite limits and colimits;

(ii) every pushout square is also a pullback square.

Proof. See [32, Proposition 1.1.3.4]. □

In any pointed ∞-category $\mathscr{C}$ which has all finite limits and colimits one has the following functors:

$$\begin{aligned}\Omega_{\mathscr{C}} : \mathscr{C} &\to \mathscr{C} \\ X &\mapsto 0 \times_X 0,\end{aligned} \quad \text{and} \quad \begin{aligned}\Sigma_{\mathscr{C}} : \mathscr{C} &\to \mathscr{C} \\ X &\mapsto 0 \sqcup_X 0.\end{aligned} \tag{3.5}$$

Notation 3.2.3. Sometimes the loop functor $\Omega_{\mathscr{C}}$ is denoted by

$$X[-1] := \Omega_{\mathscr{C}}(X)$$

and the suspension functor $\Sigma_{\mathscr{C}}$ is denoted by

$$X[1] := \Sigma_{\mathscr{C}}(X).$$

The following is a useful result to check stability of the category of spectra

Proposition 3.2.4. *Let $\mathscr{C}$ be a pointed ∞-category. Then the following are equivalent:*

(i) $\mathscr{C}$ is stable;

(ii) $\mathscr{C}$ admits finite limits and $\Omega_{\mathscr{C}}$ is an equivalence;

(iii) $\mathscr{C}$ admits finite colimits and $\Sigma_{\mathscr{C}}$ is an equivalence.

Proof. See Proposition 1.4.2.11 and Corollary 1.4.2.27 in [32]. □

Example 3.2.5.

1. The category Spctr defined in Example 2.3.15 is stable. Since finite colimits commute with filtered limits, it is not hard to argue that the category Spctr has finite colimits since Spc has finite colimits. That Spctr has finite limits is clear. Finally, by definition one has that Ω_{Spctr} is an equivalence of category, thus by Proposition 3.2.4 one has that Spctr is stable.

2. Given A an ordinary abelian category the derived ∞-category $\mathscr{D}(\mathsf{A})$ is stable. The argument is roughly as follows. Firstly one checks that $\mathscr{D}(\mathsf{A})$ admits finite colimits, this is done by considering a simplicial category A_Δ whose homotopy coherent nerve represents $\mathscr{D}(\mathsf{A})$ (see [32, Proposition 1.3.1.17]), then checking that A admits finite homotopy colimits, which implies that $\mathrm{N}^{\mathrm{h.c.}}(\mathsf{A}_\Delta) \simeq \mathscr{D}(\mathsf{A})$ will have finite colimits (see [33, Theorem 4.2.4.1]). Secondly, one checks that the suspension functor $\Sigma_{\mathscr{D}(\mathsf{A})}$, given by the homotopy pushout, is isomorphic to the shift functor. Thirdly, since the shift functor is clearly an equivalence the result follows from Proposition 3.2.4.

The next proposition compiles a number of properties of stable ∞-categories that exemplify their usefulness.

Proposition 3.2.6. *Let $\mathscr{C}$ be a stable ∞-category, then its homotopy category $\mathrm{h}\mathscr{C}$ is a triangulated category.*

Idea. A sequence $X \to Y \to Z$ in $\mathrm{h}\mathscr{C}$ is a distinguished triangle if it is the image of a (co)fiber sequence in $\mathscr{C}$. Given a distinguished triangle $X \to Y \to Z$ the map $Z \to X[1]$ comes from considering

$$0 \underset{Z}{\times} 0 \to 0 \underset{Z}{\times} Y.$$

See [32, Theorem 1.1.2.14] for the rest of the proof. □

One of the problems that stable ∞-categories solve is the functoriality of the cone construction.

Lemma 3.2.7. *Let $\mathscr{C}$ be a stable ∞-category, then one has a functor*

$$\mathit{Cofib} : \mathit{Fun}([1], \mathscr{C}) \to \mathscr{C}$$

which sends $f : X \to Y$ to an object Z such that

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

is a pushout square.

Proof. Let $f : [1] \to \mathscr{C}$ represent a morphism in $\mathscr{C}$. We denote by L_0^2 be the following category

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \\ 2 & & \end{array}$$

and consider $\imath : [1] \to L_0^2$ the inclusion of $0 \to 1$. We claim that

$$\mathrm{RKE}_\imath(f) : L_0^2 \to \mathscr{C}$$

exists and moreover that $\mathrm{RKE}_\imath(f)|_2 \simeq 0_{\mathscr{C}}$. Indeed, by Lemma 2.3.17 we only need to check that for every $X \in L_0^2$ the limit

$$\lim_{[1] \underset{[1]}{\times} (L_0^2)^{X/}} f$$

exists. Notice

- $[1] \underset{[1]}{\times} (L_0^2)^{0/} \simeq [1]$ and $\lim_{[1]} f \simeq \mathrm{ev}_0 \circ f$;
- $[1] \underset{[1]}{\times} (L_0^2)^{1/} \simeq \{1\}$ and $\lim_{\{1\}} f \simeq \mathrm{ev}_1 \circ f$;
- $[1] \underset{[1]}{\times} (L_0^2)^{2/} \simeq \emptyset$ and $\lim_{\emptyset} f \simeq 0_{\mathscr{C}}$, since by definition the limit of the empty diagram is the final object of $\mathscr{C}$[11];

Now consider $\jmath : L_0^2 \hookrightarrow [1] \times [1]$ where $[1] \times [1]$ represents the following category

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 2 & \longrightarrow & 3 \end{array}$$

[11] Is this true? Check it using the definition of limit that I gave.

where the diagram is required to commute, and the map $\jmath$ is the natural inclusion. Let $\tilde{f} := \mathrm{RKE}_\imath(f)$, we claim that the left Kan extension $\mathrm{LKE}_\jmath(\tilde{f})$ exists. Indeed, again by Lemma 2.3.17 we only need to check that certain colimits exists. The only non-trivial case is

$$L_0^2 \underset{L_0^2}{\times} ([1]\times[1])^{/3} \simeq L_0^2 \quad \text{and} \quad \mathrm{colim}_{L_0^2}\tilde{f},$$

whose existence by definition is the requirement that the diagram

$$\begin{array}{ccc} \mathrm{ev}_0(f) & \longrightarrow & \mathrm{ev}_1(f) \\ \downarrow & & \\ 0_{\mathscr{C}} & & \end{array}$$

has a pushout. Since we require that all finite colimits exist in $\mathscr{C}$ this pushout exist. So we get a functor

$$\mathrm{LKE}_\jmath \circ \mathrm{RKE}_\imath(f) : [1]\times[1] \to \mathscr{C}.$$

Finally, we let $\mathrm{Cofib} : \mathrm{Fun}([1],\mathscr{C}) \to \mathscr{C}$ be given by the composite

$$\mathrm{Fun}([1],\mathscr{C}) \overset{\mathrm{RKE}_\imath}{\to} \mathrm{Fun}(L_0^2,\mathscr{C}) \overset{\mathrm{LKE}_\jmath}{\to} \mathrm{Fun}([1]\times[1],\mathscr{C}) \overset{\mathrm{ev}_3}{\to} \mathscr{C},$$

where the last map evaluates $\mathrm{LKE}_\jmath \circ \mathrm{RKE}_\imath(f)$ on $3 \in [1]\times[1]$. □

The following diagrams are useful to prove the existence of cones in stable ∞-categories.

Existence of cofibers in stable categories I will follow the argument in these notes. The following resources are also useful for this: Harpaz's and Proudfoot's notes.

First diagram to consider:

$$\begin{array}{ccc} \Delta^1 & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \Lambda_0^2 & & \end{array}$$

Second:

$$\begin{array}{ccc} \Delta^1_{X/} & \xrightarrow{F_C} & \mathcal{C} \\ \downarrow & \nearrow & \\ (\Delta^1_{X/})^{\triangleleft} & & \end{array} \quad \text{for } X \in \Lambda_0^2, \text{ labeled as} \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

Third:

$$\begin{array}{ccc} \Lambda_0^2 & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^1\times\Delta^1 & & \end{array}$$

last:

$$\mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathrm{Fun}(\Delta^1,\mathcal{C})$$

The stable ∞-category of spectra

The most important and prototypical example of a stable ∞-category is that of spectra. Despite the definition of Example 2.3.15 one can also approach it from a dual perspective.

Let $\mathrm{Spc}_*^{\mathrm{fin}}$ denote the ∞-category of finite pointed spaces, i.e. its the ∞-category obtained from the category with a single object by adjoint all *finite* colimits. By construction this category has an endofunctor

$$\Sigma : \mathrm{Spc}_*^{\mathrm{fin}} \to \mathrm{Spc}_*^{\mathrm{fin}},$$

which satisfies conditions (a) and (b) in the definition of stable categories. However it fails condition (c). One can then consider

$$\mathrm{Spctr}^{\mathrm{fin}} := \mathrm{colim}_{\mathbb{Z}} \left(\cdots \xrightarrow{\Sigma} \mathrm{Spc}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathrm{Spc}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \cdots \right).$$

This is the stable ∞-category of *finite spectra*. One then has

Lemma 3.2.8. *The ind-completion of $Spctr^{\mathrm{fin}}$ is the category of spectra, i.e.*

$$Spctr \simeq Ind(Spctr^{\mathrm{fin}}).$$

We end this section by introducing a useful notation for stable ∞-categories.

Notation 3.2.9. For $\mathscr{C}$ a stable ∞-category, given any two objects X and Y in $\mathscr{C}$ for any $n \in \mathbb{Z}$ we define

$$\mathrm{Ext}^n_{\mathscr{C}}(X, Y) := \mathrm{Hom}_{\mathrm{h}\,\mathscr{C}}(X[-n], Y).$$

We notice that for $n < 0$ one has $\mathrm{Ext}^n_{\mathscr{C}}(X, Y) \simeq \pi_{-n}\mathrm{Hom}_{\mathscr{C}}(X, Y)$.

3.2.2 t-structures

Stable ∞-categories should be thought as the analogues of triangulated categories, or rather the coherent notion in ∞-categories that explains what the somewhat arcane axioms of a triangulated category were trying to capture.

A reasonable question is to ask what type of ∞-categories play the role of abelian 1-categories. Roughly speaking the answer are prestable ∞-categories. The notion of a prestable ∞-category is closely related to the notion of t-structure, so we start this section by defining a t-structure and then collecting some results about them and after that we give the definition of a prestable ∞-category and some results about them. We should also mention that the notion of t-structure is very useful when trying to obtain a 1-categorical statement from an ∞-categorical statement. In the theory of ∞-categories the somewhat unique way to specialize statements to discrete categories is by passing to homotopy groups, which is an inherently nonlinear procedure. The context of stable ∞-categories with a t-structure allow one to perform this operation by passing to (co-)homology groups which are usual easier to understand than homotopy groups.

Definition 3.2.10. Given $\mathscr{C}$ a stable ∞-category a *t-structure* is the data of a pair of subcategories $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ satisfying the conditions

(i) for every $X \in \mathscr{C}^{\leq 0}$ and $Y \in \mathscr{C}^{\geq 1}$ one has[12]

$$\pi_0\left(\mathrm{Hom}_{\mathscr{C}}(X, Y)\right) \simeq 0;$$

(ii) one has $\mathscr{C}^{\leq 0} \subset \mathscr{C}^{\leq 0}[-1]$ and $\mathscr{C}^{\geq 0}[-1] \subset \mathscr{C}^{\geq 0}$;

(iii) for every $X \in \mathscr{C}$ one has a fiber sequence

$$X_{\mathrm{con.}} \to X \to X_{\mathrm{cocon.}}$$

where $X_{\mathrm{cocon.}} \in \mathscr{C}^{\leq 0}$ and $X_{\mathrm{con.}} \in \mathscr{C}^{\geq 1}$.

[12] Informally, cohomological convention is "counter" going up.

Remark 3.2.11. We follow the cohomological convention as used in [6], notice that this is different than the convetion of [32, Definition 1.?.?.?].

We notice that for any n one has truncation functors

- $\tau^{\leq n} : \mathscr{C} \to \mathscr{C}^{\leq n}$ which is *right adjoint* to the natural inclusion $\mathscr{C}^{\leq n} \hookrightarrow \mathscr{C}$, since this inclusion preserves all colimits that exist in $\mathscr{C}^{\leq n}$, given any $X \in \mathscr{C}$ we have
$$\tau^{\leq n} X \to X;$$
- $\tau^{\geq n} : \mathscr{C} \to \mathscr{C}^{\geq n}$ which is *left adjoint* to the natural inclusion $\mathscr{C}^{\geq n} \hookrightarrow \mathscr{C}$, since this inclusion preserves all limits that exist in $\mathscr{C}^{\geq n}$, given any $X \in \mathscr{C}$ we have
$$X \to \tau^{\geq n} X.$$

More properties of t-structure, write as needed.

3.3 Enriched ∞-categories

In this section we give briefly sketch how one can formalize the definition of an enriched category over a monoidal category to the context of ∞-categories. We begin with the following analogue of **fc**-categories for ∞-categories.

Definition 3.3.1. An

3.4 Presentable ∞-categories

Proposition 3.4.1. *Consider $G : \mathscr{C} \to \mathscr{D}$ a functor between presentable ∞-categories and assume that*

- *G preserves limits;*
- *G preserves sifted colimits;*
- *G is conservative,*

then

(i) there exists a left adjoint $F : \mathscr{D} \to \mathscr{C}$;

(ii) F takes compact objects to compact objects;

(iii) given a set $\{D_i\}_I$ of compact projective generators of $\mathscr{D}$, then $\{F(D_i)\}_I$ is a set of compact projective generators of $\mathscr{C}$;

(iv) $\mathscr{D}$ is projectively generated implies that $\mathscr{C}$ is projectively generated.

Chapter 4

Affine schemes

Plan:

1. review the three models for derived rings: SCRs, cdgas and connective E_∞-spectra;

2. Discussion of modules over a derived rings. Definition, concept of (almost) perfect modules. What perfect modules are over an underived ring.

3. Étale, smooth and flat morphisms of derived rings. Fppf descent using t-structure and reducing to the fppf descent of the heart.

4.1 Models for affine schemes

4.1.1 Simplicial commutative rings

Let Poly denote the ordinary category of finitely generated polynomials algebras over k, i.e. its objects are algebra of the form: $k[x_1, \dots, x_n]$. We recall that given any ∞-category $\mathscr{C}$ with finite products we can define

$$\mathscr{P}_\Sigma(\mathscr{C}) := \{F : \mathscr{C}^{\mathrm{op}} \to \mathrm{Spc} \mid F \text{ preserves finite limits } \}.$$

The category $\mathscr{P}_\Sigma(\mathscr{C})$ gives a concrete way to realize the *sifted completion of* $\mathscr{C}$, i.e. it formally adjoint all sifted colimits to $\mathscr{C}$.

Remark 4.1.1. Recall that an ∞-category (or simplicial set) K is said to be *sifted* if the diagonal map

$$\delta : K \to K \times K$$

is cofinal[1]. The notion of a sifted ∞-category is motivated by understanding over what type of diagrams K colimits indexed by K in a category of objects with an algebraic structure, e.g. groups or commutative rings, can be computed as colimits of the underlying sets (or spaces). See the beginning of §5.5.8 in [33] for a nice discussion of this. The main examples to keep in mind of sifted diagrams are:

(i) any filtered ordinary category, or filtered ∞-category;

(ii) the simplicial category Δ^{op}.

Definition 4.1.2. The ∞-category of *simplicial commutative rings* is the sifted completion of the *discrete* category of polynomial rings, i.e.

$$\mathrm{SCR} := \mathscr{P}_\Sigma(\mathsf{Poly}^{\mathrm{op}}).$$

[1] I.e. for any functor $F : K \times K \to \mathrm{Spc}$ one has an isomorphism

$$\mathrm{colim}_{K\times K} F \xrightarrow{\simeq} \mathrm{colim}_K F \circ \delta.$$

Remark 4.1.3. One can prove that the category SCR is equivalent to the ∞-category underlying the simplicial model category of simplicial commutative rings with the projective model structure. Indeed, by [33, Corollary 5.5.9.3] one has an equivalence

$$\mathscr{P}_{\Sigma}(\mathsf{Poly}^{\mathrm{op}}) \simeq \mathrm{N}^{\mathrm{hc}}(\mathsf{SCR}_{\mathrm{cf}})$$

where SCR denotes the ordinary category of finite product preserving functors from $\mathsf{Poly}^{\mathrm{op}}$ to Set_{Δ}[2] endowed with the (simplicial) model where weak equivalences and fibrations are those which give weak equivalences and fibrations when evaluated at any object $C \in \mathsf{Poly}^{\mathrm{op}}$[3]. The subscript cf as usual means that we are considering the subcategory of cofibrant-fibrant objects.

We record here for the reader's convenience a couple of properties of the category SCR, which essentially are general properties of the sifted completion of any category (see [33, §5.5.8]).

Proposition 4.1.4. *(i) Given an ∞-category $\mathscr{D}$ which admits sifted colimits one has an equivalence*

$$\mathit{Fun}(\mathit{SCR}, \mathscr{D}) \xrightarrow{\simeq} \mathit{Fun}_{\Sigma}(\mathsf{Poly}, \mathscr{D})$$

where the subscript on the right-hand side means we restrict to functors that preserve sifted colimits;

(ii) The category SCR is presentable and has all colimits;

(iii) The natural inclusion $\imath : \mathsf{Poly} \hookrightarrow \mathit{SCR}$ preserves coproducts;

(iv) The essential image of $\imath$ consist of compact and projective objects of SCR, which generate it under sifted colimits.

4.1.2 Connective differential graded algebras

Let k be a fixed field and consider the derived ∞-category associated to the abelian category of vector spaces over k (see Example 2.2.49). This category has a t-structure, so we let $\mathrm{Vect}_k^{\leq 0}$ denote the subcategory of connective objects.

Definition 4.1.5. Let

$$\mathrm{CAlg}_k := \mathrm{CAlg}(\mathrm{Vect}_k^{\leq 0})$$

denote the ∞-category of commutative algebra objects[4] in $\mathrm{Vect}_k^{\leq 0}$.

Remark 4.1.6. Definition 4.1.5 is justified by noticing that the category of discrete objects of CAlg_k identifies with the ordinary category of commutative k-algebras. Indeed, by [32, Proposition 7.1.3.15] the discrete objects of CAlg_k are the same as commutative algebra objects in the heart of $\mathrm{Vect}_k^{\leq 0}$, which by construction is the usual ordinary category of vector spaces over k, on which the notion of a commutative algebra objects recovers the classical notion (see [32, Remark 7.1.3.16][5]).

[2]It is not hard to see this is equivalent to the ordinary category of simplicial commutative rings

[3]The reader which is interested can make the description of this model structure more explicitly in terms of simplicial commutative rings, we claim that this will give the so-called projective model structure.

[4]Formally defined as maps of ∞-operads:

$$\begin{array}{ccc} \mathrm{Fin}_* & \longrightarrow & \mathrm{Vect}_k^{\leq 0, \mathrm{Fin}_*} \\ & \searrow \quad \swarrow & \\ & \mathrm{Fin}_* & \end{array}$$

where $\mathrm{Vect}_k^{\leq 0,\mathrm{Fin}_*} \to \mathrm{Fin}_*$ is the *Cartesian* fibration encoding the symmetric monoidal structure of $\mathrm{Vect}_k^{\leq 0}$. More concretely, this is equivalent to the data of a functor $A : \mathrm{Fin}_* \to \mathrm{Vect}_k^{\leq 0}$ satisfying the condition that for all every $n \geq 1$ one has an equivalence (see [32, Proposition 2.4.2.5])

$$A([n]) \to \prod_{i=1}^{n} A([1]),$$

where the map is induced by that map $[n] \to [1]^{\times n}$ obtained by the product of the maps that send $i \mapsto 1$ and everything else to 0.

[5]Notice that in our set up k is discrete, i.e. $k \simeq \pi_0(k)$.

4.1.3 Connective $\mathbb{E}_\infty$-algebras

Let Spctr denote the stable ∞-category of spectra (see Example 2.3.15), this category has a t-structure determined by

$$\mathrm{Spctr}^{\leq 0} := \{S \in \mathrm{Spctr} \mid \pi_{-k}(S) = 0\ \forall k > 0\} \text{ and } \mathrm{Spctr}^{\geq 0} := \{S \in \mathrm{Spctr} \mid \pi_{-k}(S) = 0\ \forall k < 0\}\,[6].$$

It is easy to see that the symmetric monoidal structure of Spctr restricts to a symmetric monoidal structure on $\mathrm{Spctr}^{\leq 0}$[7]. Another option for the category of derived affine schemes is to take

Definition 4.1.7. Let the category of $\mathbb{E}_\infty$*-rings* be

$$\mathrm{Alg}_{\mathbb{E}_\infty} := \mathrm{CAlg}(\mathrm{Spctr}^{\leq 0}),$$

i.e. the category of commutative algebra objects in the category of connective spectra.

We could also have considered commutative algebra objects in nonconnective spectra, i.e. $\mathrm{Alg}^{\mathrm{nc}}_{\mathbb{E}_\infty} := \mathrm{CAlg}(\mathrm{Spctr})$.

Remark 4.1.8. Notice that in Definition 4.1.7 the notion of commutative algebra object is that of a homotopy coherent commutative multiplication. One can also consider a relative version of Definition 4.1.7 by considering an object R in $\mathrm{Alg}_{\mathbb{E}_\infty}$ and taking the comma category $\mathrm{Alg}_{\mathbb{E}_\infty,A} := (\mathrm{Alg}_{\mathbb{E}_\infty})_{A/}$. By [32, Proposition 3.4.1.4 and Theorem 5.1.4.10] this is equivalent to considering the category $\mathrm{Mod}_A^{\leq 0}$ of connective A-modules in spectra and taking commutative algebra objects in this, i.e.

$$\mathrm{CAlg}(\mathrm{Mod}_A^{\leq 0}) \simeq \mathrm{CAlg}(\mathrm{Spctr}^{\leq 0})_{A/}$$

for any $A \in \mathrm{Alg}_{\mathbb{E}_\infty}$.

Remark 4.1.9. We define $\mathrm{Alg}^{\mathrm{disc.}}_{\mathbb{E}_\infty,A}$ the subcategory of discrete objects of $\mathrm{Alg}_{\mathbb{E}_\infty,A}$ to be that of $\mathbb{E}_\infty$-rings whose underlying spectrum is discrete, i.e. has vanishing non-zero homotopy groups. One has an equivalence

$$\pi_0 : \mathrm{Alg}^{\mathrm{disc.}}_{\mathbb{E}_\infty} \to \mathrm{CAlg}^{\mathrm{disc.}} \tag{4.1}$$

between the category of discrete $\mathbb{E}_\infty$-algebras and that of ordinary commutative rings. Indeed, the equivalence (4.1) is obtained in the same way as in Remark 4.1.6 by noticing that the discrete objects of $\mathrm{Alg}_{\mathbb{E}_\infty}$ are equivalent to the commutative algebra objects in the heart of Spctr which is equivalent to the category of abelian groups.

4.1.4 Comparison of models

Let $\imath : \mathsf{CAlg}_k \to \mathrm{CAlg}_k$ denote the inclusion of the category of discrete objects, which we identify with the ordinary category of discrete commutative rings by the Remark 4.1.6. We let $\Theta_0 : \mathsf{Poly} \to \mathrm{CAlg}_k$ denote the restriction of this functor to discrete polynomial algebras, by Proposition 4.1.4 one obtains a sifted colimit-preserving functor

$$\Theta : \mathrm{SCR} \to \mathrm{CAlg}_k. \tag{4.2}$$

Proposition 4.1.10 ([27]*Proposition 4.1.11)**.** *Assume that k is a $\mathbb{Q}$-algebra, then the functor (4.2) is an equivalence of categories.*

Proof. The proof is done by observing that Θ preserves colimits, because polynomial k-algebras are flat as k-modules, that it preserves limits (which can be computed in Spc by evaluating a simplicial commutative ring on $k[x]$), and also that it is conservative. Then one notices that the symmetric power of a flat k-module M is the same if taken in either the category of simplicial commutative rings or of cdgas, thus by using [32, Proposition 4.7.3.18] one gets that the essential image of Θ_0 consists of compact and projective generators. The general properties of sifted completion ([33, Proposition 5.5.8.25]) then give that Θ is an equivalence. □

[6]Notice that we are using the cohomological convention for indexing the t-structure, one should think of $H^k(S) := \pi_{-k}(S)$.

[7]Indeed, the inclusion $\mathrm{Spctr}^{\leq 0} \hookrightarrow \mathrm{Spctr}$ preserves colimits, and $\mathrm{Spctr}^{\leq 0}$ is generated under colimits by the sphere spectrum. The result then follows from the fact that the tensor product of Spctr preserves colimits on each variable.

Question: Can one describe the functor θ more explicitly? What is its relation with the normalization functor $N : \mathsf{SCR} \to \mathsf{CAlg}$ given by:

$$N(R_\bullet) := (...)?$$

One can also prove that CAlg_k and $\mathrm{Alg}_{\mathbb{E}_\infty,k}$ are equivalent. Here is the sketch of an argument. Consider $u : \mathrm{Spctr} \to \mathrm{Vect}$ the unit in the symmetric monoidal ∞-category of presentable stable cocomplete ∞-categories, where the tensor product is Lurie's tensor product as discussed in **??**. The functor u admits a right adjoint

$$u^{\mathrm{R}} : \mathrm{Vect} \to \mathrm{Spctr},$$

which is t-exact. Thus, restricting to connective objects one has a map $\mathrm{Vect}^{\leq 0} \to \mathrm{Spctr}^{\leq 0}$ which induces a map between commutative algebra objects:

$$\imath : \mathrm{CAlg}(\mathrm{Vect}^{\leq 0}) \to \mathrm{CAlg}(\mathrm{Spctr}^{\leq 0}).$$

Moreover, for any $A \in \mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$ the functor $\imath$ induces a map

$$\mathrm{Mod}_A(\mathrm{Vect}^{\leq 0}) \to \mathrm{Mod}_{\imath(A)}(\mathrm{Spctr}^{\leq 0}). \tag{4.3}$$

By considering the functor induced on commutative algebra objects by (4.3) we get

$$\Psi : \mathrm{CAlg}(\mathrm{Vect}_k^{\leq 0}) \simeq \mathrm{CAlg}_k \to \mathrm{Alg}_{\mathbb{E}_\infty,k} \simeq \mathrm{CAlg}(\mathrm{Mod}_{\imath(k)}(\mathrm{Spctr}^{\leq 0})).$$

Proposition 4.1.11. *Let k be a commutative ring, then the functor*

$$\Psi : CAlg_k \to Alg_{\mathbb{E}_\infty,k}$$

is an equivalence of categories.

Proof. By [32, Theorem 7.1.2.13] one has an equivalence

$$\mathrm{Vect}_k \simeq \mathrm{Mod}_{\imath(k)}(\mathrm{Spctr}).$$

□

A comment is in order regarding Proposition 4.1.11. By our very definition of the ∞-category CAlg_k there wasn't many difference between CAlg_k and $\mathrm{Alg}_{\mathbb{E}_\infty,k}$ to start with. The point is that Definition 4.1.5 is already a non-strict commutative differential graded structure on a complex of k vector spaces.

A probably more concrete object one could consider is the following. Let CDGA_k denote the ordinary category of commutative differential graded algebras over k, i.e.

- objects are cochain complexes A^* of k-vector spaces with a graded commutative (*strict*) multiplication $\mu : A^* \times A^* \to A^*$;
- morphisms are maps of cochain complexes that strictly respect the multiplication.

The category CDGA_k admits a model structure, whose weak equivalences are maps of cdgas that induce a quasi-isomorphism on the underlying cochain complexes (see [32, Proposition 7.1.4.5]). The less trivial comparison between models is the following:

Proposition 4.1.12. *Assume k contains the field of rational numbers. Let $CDGA_k$ denote the ∞-category associated to CDGA_k*[8] *then one has an equivalence*

$$CDGA_k \simeq Alg^{\mathrm{nc}}_{\mathbb{E}_\infty,k}.$$

Proof. See [32, Proposition 7.1.4.11] for a proof and that section for the full description of the model structure on CDGA_k. □

[8]This is obtained by a procedure similar to that described in Remark 2.2.63. The subtlety here is that we want to make sure that the resulting ∞-category has a (symmetric) monoidal structure (see [32, §4.1.7] for a discussion of how to do this).

4.2 Properties of affine schemes

In this section we introduce many notions that are useful to understand affine schemes a bit better.

4.2.1 Coconnectiveness

Definition 4.2.1. An affine scheme $S = \mathrm{Spec} A$ is said to be *n-coconnective* if

$$H^{-k}(A) = 0 \ \ \forall k > n.$$

In particular, a 0-coconnective affine schemes is called *discrete*.

We let ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ denote the full subcategory of $\mathrm{Sch}^{\mathrm{aff}}$ generated by the n-coconnective affine schemes. One notices that the inclusion ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$ admits a right adjoint

$${}^{\leq n}(-) : \mathrm{Sch}^{\mathrm{aff}} \to {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$$

given by sending $S = \mathrm{Spec} A$ to the spectrum of the truncation[9] $\tau^{\leq n}(S) = \mathrm{Spec}(\tau^{-n}(A))$.

We will denote by

$$\tau^{\leq n} : \mathrm{Sch}^{\mathrm{aff}} \stackrel{\tau^{\leq n}}{\to} {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}.$$

the colocalization[10] functor obtained from this adjunction.

Remark 4.2.2. For an affine scheme $S = \mathrm{Spec}(A)$ we notice that A is an n-truncated object of the ∞-category $\mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$ if and only if S is n-coconnective.

Definition 4.2.3. An affine scheme S is said to be *eventually coconnective* if $S \simeq \tau^{\leq n}(S)$ for some n. We will denote by ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ the full subcategory of eventually coconnective affine schemes.

4.2.2 Finiteness conditions

Definition 4.2.4. Given an affine scheme $S = \mathrm{Spec}(A)$ we say that

- S is of *finite type* if

  (i) $H^0(A)$ is of finite type over k;

  (ii) for all $i \in \mathbb{Z}$, $H^i(A)$ is a finitely generated $H^0(A)$-module;

  (iii) $H^i(A) = 0$ for $i \ll 0$.

  We let $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \hookrightarrow {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$ denote the corresponding subcategory.

- S is *almost of finite type* if

  (i) $H^0(A)$ is of finite type over k;

  (ii) for all $i \in \mathbb{Z}$, $H^i(A)$ is a finitely generated $H^0(A)$-module.

  We let $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$ denote the corresponding subcategory. Notice that S is almost of finite type if and only if ${}^{\leq n}S$ is of finite type[11] for all $n \geq 0$.

---

[9] Notice that in an ∞-categorical context only the non-naïve truncation, e.g. $\tau^{\geq -1}(\cdots \to A^{-2} \stackrel{d^{-2}}{\to} A^{-1} \stackrel{d^{-1}}{\to} A^{-}0 \stackrel{d^{-1}}{\to} A^0 \to \cdots) = (\cdots \to 0 \to \frac{\mathrm{Ker} d^{-1}}{\mathrm{Im} d^{-2}} \stackrel{d^{-1}}{\to} A^{-}0 \stackrel{d^{-1}}{\to} A^0 \to \cdots)$ makes sense.

[10] Recall that for an adjunction $F : \mathscr{C} \leftrightarrow \mathscr{D}$ is said to give a localization (resp. colocalization)

$$G \circ F \quad (\text{resp. } F \circ G)$$

if G is fully faithful, resp. F is fully faithful. Some references refer to F as the localization functor (resp. G as the colocalization functor).

[11] Notice that what this means is clear since the conditions (i-iii) above make sense when restricted to the subcategory ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

Exercise 4.2.5. (i) $k[\epsilon]$ with $|\epsilon| = -1$ is of finite type;

(ii) $k[\eta]$ with $|\eta| = -2$ is almost of finite type, but not of finite type.

The following result is useful when trying to understand an arbitrary affine scheme.

Proposition 4.2.6. *For any $S \in {}^{\leq n}Sch^{\mathrm{aff}}$ one has*

$$S \simeq \lim_{S \to S' \mid {}^{\leq n}Sch^{\mathrm{aff}}{}_{\mathrm{ft}}} S',$$

i.e. $Pro({}^{\leq n}Sch^{\mathrm{aff}}{}_{\mathrm{ft}}) \simeq {}^{\leq n}Sch^{\mathrm{aff}}$.

Proposition 4.2.7. *An object $R \in CAlg$ is compact and projective if and only if R is a retract of $Sym(V)$ for $V \simeq k^{\oplus n}$ for some $n \geq 0$, where*

$$CAlg \underset{Sym}{\overset{oblv}{\leftrightarrows}} Vect$$

is the forgetful-free adjunction between commutative algebras and vector spaces.

(1) Check the statement, I think V should be perfect and connective here. (2) Write an argument.
Here is an idea of how to prove Proposition 4.2.7 from my notes.

Lemma 4.2.8. *The functor $oblv : CAlg \to Vect$ satisfies the conditions of Proposition 3.4.1, namely it preserves limits, sifted colimits and is conservative.*

Corollary 4.2.9. *The object $Sym(k) \simeq k[x]$ is a compact projective generator of CAlg.*

The end of Talk 16 is trying to do something I don't quite understand.

4.2.3 Anything else?

4.3 Schemes

In this section we introduce derived schemes and prove some basic facts about them.

Chapter 5

Modules on derived rings

1. Construct dg-modules over a dg-ring as an ∞-category.
   2. Construct modules over a derived ring using the $\mathscr{L}\mathscr{M}^{\otimes}$ operad.

5.1 Model-independent formulation

5.1.1 Module categories

Let $\mathscr{C}$ be a stable monoidal ∞-category, i.e. this corresponds to a functor

$$\mathscr{C}^{\otimes} : \Delta^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\mathrm{st}} \tag{5.1}$$

satisfying:

- $\mathscr{C}^{\otimes}([0]) \simeq \mathrm{pt}$;
- for any $n \geq 1$ the map $[n] \to [1]^n$ obtained by projection onto the spine of $[n]$ gives an equivalence:

$$\mathscr{C}^{\otimes}([n]) \xrightarrow{\simeq} \mathscr{C}^{\otimes}([1])^{\times n}.$$

Consider the category Δ^+ whose:

- objects are $[n]$ and $[n]^+ := \{0 < 1 < \cdots < n < +\}$ for $n \geq 0$;
- morphisms are:

  (i) functors $[n] \to [m]$;

  (ii) functors $[n] \to [m]^+$ whose essential image does not contain $+$; and

  (iii) functors $[n]^+ \to [m]^+$ that send $+$ to $+$, and such that the pre-image of $+$ is only $+$.

Definition 5.1.1. Given a stable monoidal ∞-category $\mathscr{C}$ a category $\mathscr{M} \in \mathrm{Cat}_{\infty}^{\mathrm{st}}$ is said to be a *$\mathscr{C}$-module category* if there exists an extension of the functor (5.1) to a functor

$$(\mathscr{C}, \mathscr{M})^{\otimes} : \Delta^+ \to \mathrm{Cat}_{\infty}^{\mathrm{st}} \tag{5.2}$$

satisfying:

- for any $n \geq 0$ the morphism $\alpha_n : [n] \sqcup [0]^+ \to [n]^+$ given by $\alpha_n|_{[n]}(i) = i$ and $\alpha_n|_{[0]^+}(0) = n$ gives an equivalence

$$(\mathscr{C}, \mathscr{M})^{\otimes}([n]^+) \xrightarrow{\alpha_n^+} \mathscr{C}^{\otimes}([n]) \times (\mathscr{C}, \mathscr{M})^{\otimes}([0]^+).$$

In this situation we will often abuse notation and abbreviate all the data of (5.2) by saying that the underlying ∞-category $\mathscr{M} := (\mathscr{C}, \mathscr{M})^{\otimes}([0]^+)$ has the structure of a $\mathscr{C}$-module.

Remark 5.1.2. Let's make explicit part of the structure encoded in (5.2). The unique morphism $[0]^+ \to [1]^+$ corresponds to the action map

$$a_{\mathscr{C}} : \mathscr{C} \times \mathscr{M} \simeq (\mathscr{C}, \mathscr{M})^{\otimes}([1]^+) \to (\mathscr{C}, \mathscr{M})^{\otimes}([0]^+) \simeq \mathscr{M}.$$

The higher morphisms encode the compatibility of $a_{\mathscr{C}}$ with the monoidal structure of $\mathscr{C}$.

Remark 5.1.3. Similarly to Definition **??** suppose that $\mathscr{C}$ and $\mathscr{C}'$ are two monoidal structure and $\mathscr{M}$ and $\mathscr{M}'$ a $\mathscr{C}$-module and $\mathscr{C}'$-module categories respectively. Then we say that a functor $F : \mathscr{M} \to \mathscr{M}'$ is *compatible* (resp. *right-lax compatible*, *left-lax compatible*) [1] with the actions of $\mathscr{C}$ and $\mathscr{C}'$, or a functor from $\mathscr{C}$-modules to $\mathscr{C}'$-modules if there exists a map[2]

$$F^{\otimes,\Delta^{+,\mathrm{op}}} : (\mathscr{C}, \mathscr{M})^{\otimes,\Delta^{+,\mathrm{op}}} \to (\mathscr{C}', \mathscr{M}')^{\otimes,\Delta^{+,\mathrm{op}}}$$

over $\Delta^{+,\mathrm{op}}$ which takes any (resp. Make these morphisms explicit.) coCartesian morphism in $(\mathscr{C}, \mathscr{M})^{\otimes,\Delta^{+,\mathrm{op}}}$ to a coCartesian morphism in $(\mathscr{C}', \mathscr{M}')^{\otimes,\Delta^{+,\mathrm{op}}}$.

Example 5.1.4.

(i) Let $\mathscr{C}$ be a monoidal category, consider $\jmath : \Delta^+ \to \Delta$ given by $\jmath([n]) = [n]$ and $\jmath([n]^+) = [n+1]$, then one has that

$$\jmath^*\mathscr{C}^{\otimes} : \Delta^+ \to \mathrm{Cat}_{\infty}^{\mathrm{st}}$$

endows $\mathscr{C}$ with the structure of a $\mathscr{C}$-module. Check this! In particular, one has that Vect, Spc, and Spctr are symmetric monoidal categories.

(ii) Let Vect denote the category of vector spaces over k, a *DG category* $\mathscr{C}$ is a stable ∞-category with a Vect-module structure. More interestingly, for any $A \in \mathrm{CAlg}$ the category Mod_A (defined below) is a DG category.

(iii) Give another example.

To formally write down the category of $\mathscr{C}$-modules for $\mathscr{C}$ a monoidal category we need to introduce some ambient ∞-categories of monoidal categories and module categories.

Definition 5.1.5. Let $\mathrm{Cat}_{\infty}^{\mathrm{Mon}} \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}_{\infty})$ denote the subcategory whose:

- objects are functors encoding the data of monoidal category, i.e. satisfying the conditions of Definition 3.1.14;
- morphisms are natural transformations encoding right-lax monoidal functor (see Definition **??**.)

Similarly, we let $\mathrm{Cat}_{\infty}^{\mathrm{Mon,Mod}} \subset \mathrm{Fun}(\Delta^{+,\mathrm{op}})$ denote the subcategory whose:

- objects are functors encoding of a module category, i.e. satisfying the conditions of Definition 5.1.1;
- morphisms are natural transformations encoding right-lax compatible functors (see Remark 5.1.3.)

Definition 5.1.6. Given a monoidal category $\mathscr{C}$ the category of $\mathscr{C}$-modules is defined as

$$\mathscr{C}-\mathrm{Mod} := \{\mathscr{C}^{\otimes}\} \underset{\mathrm{Cat}_{\infty}^{\mathrm{Mon,Mod}}}{\times} \mathrm{Cat}_{\infty}^{\mathrm{Mon}}.$$

Example 5.1.7. The category of DG-categories is

$$\mathrm{DGCat} = \mathrm{Vect} - \mathrm{Mod}.$$

Similarly, to how we defined the category of $\mathscr{C}$-module category to define the category of modules for a fixed associative algebra object A in a monoidal category $\mathscr{C}$ we first need to set up a general piece of data which encodes pairs of an associative algebra object and a module for it.

[1] Though this is not a condition, but rather extra data.

[2] Here $(\mathscr{C}, \mathscr{M})^{\otimes,\Delta^{+,\mathrm{op}}} \to \Delta^{+,\mathrm{op}}$ denotes the coCartesian fibration associated to the functor (5.2).

Definition 5.1.8. Given $(\mathscr{C},\mathscr{M})^{\otimes} \in \mathrm{Cat}_{\infty}^{\mathrm{Mon,Mod}}$ a pair of a monoidal category $\mathscr{C}$ and a $\mathscr{C}$-module $\mathscr{M}$, the data of a pair (A, M) where A is an associative algebra object of $\mathscr{C}$ and M is an A-module is a right-lax compatible map

$$(A, M) : *^{\Delta^{+},\mathrm{op}} \to (\mathscr{C},\mathscr{M})^{\otimes,\Delta^{+}}.$$

We will denote the category of such as $\mathrm{mod}AssocAlg{+}\mathrm{mod}(\mathscr{C},\mathscr{M})$. Do I need to specify what the morphisms in this category are?

Definition 5.1.9. Consider $\mathscr{C}$ a monoidal category, $\mathscr{M}$ a $\mathscr{C}$-module, and $A \in \mathscr{C}$ an associative algebra object. The category of *left A-modules in $\mathscr{M}$* is defined as

$$A - \mathrm{mod}^{\mathrm{left}}(\mathscr{M}).$$

Remark 5.1.10. Definition 5.1.9 has a clear variant where one considers right modules, for that one needs to consider pairs $(\mathscr{M},\mathscr{C})$ where $\mathscr{C}$ is a monoidal category and $\mathscr{M}$ is a $\mathscr{C}$-module category.

Notation 5.1.11. In the context of Definition 5.1.9 in most of this text $\mathscr{C}$ will actually be a symmetric monoidal category and A will be a commutative algebra object. In this case we will drop the superscript left from the notation and simply write

$$A - \mathrm{mod}(\mathscr{M}).$$

We will often encounter the situation where $\mathscr{M} = \mathrm{Vect}$, in this case we will drop it from notation and simply write $A - \mathrm{mod}$ for $A - \mathrm{mod}(\mathrm{Vect})$.

Example 5.1.12. Recall that for us $\mathrm{CAlg} = \mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$. We also notice that Vect is a $\mathrm{Vect}^{\leq 0}$-module category, by restricting the structure of Vect-module on Vect to $\mathrm{Vect}^{\leq 0}$. Given $R \in \mathrm{CAlg}$ we define the category of R-modules to be

$$R - \mathrm{mod} := R - \mathrm{mod}(\mathrm{Vect}).$$

5.1.2 Tensor structure

We give a quick discussion of the tensor structure in the category Mod_A, when A is a derived ring.

Given A and B associative algebras objects in a symmetric monoidal ∞-category $\mathscr{C}$, we let ${}_A\mathrm{Mod}_B$ denote the ∞-category of (A, B)-bimodules. Informally speaking this can be formalized by consider the category $\Delta^{+,+',\mathrm{op}}$ whose objects are the linearly ordered sets $[n]^{+,+'} := \{+' < 0 < 1 < \cdots < n < +\}$ and morphisms are defined similarly to those of $\Delta^{+,\mathrm{op}}$ and then considering functors

$$\Delta^{+,+',\mathrm{op}} \to \mathscr{C}$$

which take $+'$ to A and $+$ to B[3] The following is Theorem 4.4.2.8 in [32]:

Theorem 5.1.13. *Given three associative algebra objects A, B and C in a monoidal ∞-category $\mathscr{C}$ one has a functor*

$$(-) \otimes_B (-) : {}_A Mod_B \times {}_B Mod_C \to {}_A Mod_C \tag{5.3}$$

which is uniquely (up to a contractible space of choices) characterized by:

(i) *for any $N \in$ and $M \in$ the tensor product is given by the Bar construction*

$$N \otimes_B M \simeq \text{Figure out tikz thing here!} colim(\cdots \to N \otimes B \otimes M \to N \otimes M)$$

(ii) *the functor $(-) \otimes_B (-)$ preserves geometric realizations, i.e. colimits indexed by Δ^{op}, on each variable.*

We notice that in the case that A is a commutative algebra object in a symmetric monoidal ∞-category $\mathscr{C}$ one has an equivalence (give a reference for this!)

$$\mathrm{Mod}_A \simeq {}_A\mathrm{Mod}_A.$$

Corollary 5.1.14. *Let $\mathscr{C}$ be a symmetric monoidal category and $A \in CAlg(\mathscr{C})$, then the category Mod_A inherits a symmetric monodial structure whose map $Mod_A \times Mod_A \to Mod_A$ underlying $[2] \to [1]$ is given by (5.3).*

[3]Strictly speaking, this describes how to consider (A, B)-modules with value in $\mathscr{C}$ itself, seen as a bimodule category over $\mathscr{C}$. The more general situation where we allow for an ∞-category $\mathscr{M}$ which is a left and right module category over $\mathscr{C}$ needs to be formalized in two steps as we explained in §**??**.

5.1.3 Base change

Let $f : A \to B$ be a morphism in CAlg, then one has a canonical functor

$$\mathrm{oblv} : \mathrm{Mod}_B \to \mathrm{Mod}_A$$

given informally by forgetting the B-module structure to an A-module structure. Explain how this is defined more formally.

Proposition 5.1.15. *The functor oblv has a left adjoint $(-) \otimes_A B : Mod_A \to Mod_B$, where B is considered as an A-module via oblv and we abuse notation and write $(-) \otimes_A B$ for the functor*

$$Mod_A \times Mod_B \overset{id_{Mod_A} \times oblv'}{\to} {}_A Mod_A \times {}_A Mod_B \overset{(-)\otimes_A B}{\to} {}_A Mod_B \to Mod_B.$$

5.1.4 Compatibility

First we notice that since Vect is a stable category and Spctr is the unit object in $\mathrm{Cat}_\infty^{\mathrm{st}}$ then Vect has the structure of a Spctr-module. Moreover, this restricts to a $\mathrm{Spctr}^{\leq 0}$-module structure on Vect.

Second the right-lax monoidal functor $u^{\mathrm{R}} : \mathrm{Vect} \to \mathrm{Spctr}$ is compatible with the structure of $\mathrm{Spctr}^{\leq 0}$-module of these categories.

Also recall that one has a comparison functor (see §4.1.4)

$$\Psi : \mathrm{CAlg} \to \mathbb{E}_\infty \mathrm{Alg}.$$

Now given $R \in \mathrm{CAlg}$ one has a couple of options for the category of R-modules:

- $R - \mathrm{mod}(\mathrm{Vect})$;
- $\Psi(R) - \mathrm{mod}(\mathrm{Vect})$;
- $\Psi(R) - \mathrm{mod}(\mathrm{Spctr})$.

One has natural maps:

$$R - \mathrm{mod}(\mathrm{Vect}) \overset{\Psi^*}{\to} \Psi(R) - \mathrm{mod}(\mathrm{Vect}) \overset{\mathrm{DK}}{\to} \Psi(R) - \mathrm{mod}(\mathrm{Spctr}).$$

We claim that Ψ^* is an equivalence and DK is fully faithful.

Figure out a good name for the functors above.

Give an argument for this, starting with explicitly getting these functors. **Question:** Is it the case that Ψ is an equivalence?

Maybe this part comes first in this section. Also one might want to split this section.

Let $\mathrm{CAlg}^{\mathrm{disc.}}$ denote the subcategory of CAlg consisting of its 0-truncated objects (see **??**). One has an equivalence[4]

$$\mathrm{CAlg}^{\mathrm{disc.}} \simeq \mathrm{CAlg}^{\tau_{\geq 0}}(\mathrm{Vect}^{-}) \simeq \mathrm{CAlg}(\mathrm{Vect}^{\heartsuit}).$$

Let $\imath : \mathrm{CAlg}^{\mathrm{disc.}} \hookrightarrow \mathrm{CAlg}$ denote the natural inclusion. For any ordinary commutative ring R one can consider $\imath(R) - \mathrm{mod}$ the ∞-category of modules over R as a particular case of any of the construction above.

Also recall that in Example 2.2.50 we constructed the ∞-category of k vector spaces for k a field. A sanity check result is that these two constructions agree as symmetric monoidal ∞-categories. This is proved in two steps. The first relies in the following general result which characterizes which stable ∞-categories are equivalent to a category of modules over some derived ring

Theorem 5.1.16. *Given $\mathscr{C}$ a stable ∞-category, then*

$$\mathscr{C} \simeq A - mod^{\mathrm{right}}$$

for some $A \in AAlg(Spctr)$ if and only if:

[4] Moreover, the category $\mathrm{CAlg}(\mathrm{Vect}^{\heartsuit})$ is the ∞-category corresponding to the ordinary category CAlg.

(i) $\mathscr{C}$ *is presentable;*

(ii) there exists $X \in \mathscr{C}$ *a compact object that generates* $\mathscr{C}$*, i.e. for all* $Y \in \mathscr{C}$ *one has*

$$Ext^n_{\mathscr{C}}(X, Y) = 0 \quad \forall n \in \mathbb{Z} \quad \Rightarrow \quad Y \simeq 0.$$

Moreover, one can describe A *explicitly,*

$$A \simeq End_{\mathscr{C}}(X),$$

where $End_{\mathscr{C}}(X)$ *denotes the* $\mathbb{E}_1$*-object in spectra obtained from the enrichment of* $\mathscr{C}$ *in spectra.*

Fill the section about how to obtain an enriched endomorphism object and refer to it here.

Remark 5.1.17. The $\mathbb{E}_1$-algebra A in the statement of the theorem above is not unique it actually depends on the choice of compact generator X.

We can now apply Theorem 5.1.16 to our situation to obtain the following sanity check.

Proposition 5.1.18. *Let* k *be an ordinary commutative ring, i.e.* $R \simeq H^0(R)$*, then one has an equivalence of categories*

$$\mathscr{D}(R) \simeq Mod_R,$$

where $\mathscr{D}(R)$ *is the derived* ∞*-category associated to* R *(see Example 2.2.49) and* Mod_R *is the* ∞*-category associated to* R *seen as a derived ring (see Definition 5.1.9).*

Proof. First, we notice that $R \in \mathscr{D}(R)$ is a compact generator. Indeed, for any $M \in \mathscr{D}(R)$ if

$$\mathrm{Ext}^n(R, M) \simeq H^n(M) \simeq 0$$

for all $n \in \mathbb{Z}$, then $M \simeq 0$ by definition of $\mathscr{D}(R)$. Since R is self-dual one has that $\mathrm{Hom}_{\mathscr{D}(R)}(R, -) \simeq R \otimes (-)$ commutes with filtered colimits, hence R is compact. Now, let $A := \mathrm{End}_{\mathscr{D}(R)}(R)$, since we have

$$\mathrm{Ext}^i(R, R) \simeq \begin{cases} H^0(R) & \text{if } i = 0; \\ 0 & \text{else.} \end{cases}$$

Since R is classical, one obtains that $\mathscr{D}(R) \simeq \mathrm{Mod}_R$. □

Warning 5.1.19. The above argument only shows that $\mathscr{D}(R) \simeq \mathrm{Mod}_R$ are equivalent as ∞-categories, but not as *symmetric monoidal* ∞-categories. For that one needs to be a bit more careful, see [32]*Proposition 7.1.2.7.

5.2 A brief study of R-modules

It will be useful for us to consider the following subcategories of Mod_R.

$$\mathrm{Perf}_R \hookrightarrow \mathrm{APerf}_R \hookrightarrow \mathrm{Mod}_R^- \hookrightarrow \mathrm{Mod}_R$$

defined as follow:

- the subcategory of *perfect* R*-modules* Perf_R can be equivalently defined as:

$$\mathrm{Perf}_R \simeq \langle R \rangle_{\oplus, \mathrm{id}} \simeq \{ \text{ perfect objects of } \mathrm{Mod}_R \} \simeq \{ \text{ dualizable objects of } \mathrm{Mod}_R \},$$

  where $\langle R \rangle$ denotes the smallest stable subcategory of Mod_R containing R and stable under direct sums and retracts;

- the subcategory of *almost perfect* R*-modules* APerf_R is defined as: $M \in \mathrm{Mod}_R$ such that (i) $M \in \mathrm{Mod}_R^-$ and (ii) $\tau^{\geq -n}(M)$ is compact in $\mathrm{Mod}_R^{\geq -k}$ for every $k \geq 0$;

- the subcategory of eventually connective objects Mod_R^- is defined as:

$$\mathrm{Mod}_R^- := \bigcup_{k\geq 0} \mathrm{Mod}_R^{\leq n}.$$

Remark 5.2.1. In the equivalent descriptions of perfect R-modules, the first equivalence is [32]*7.2.4.2, which together with [33]*§5.3 implies that $\mathrm{Mod}_R \simeq \mathrm{Ind}(\mathrm{Perf}_R)$ and the second equivalence is [32]*7.2.4.4.

We recall that we are assuming that $R \in \mathrm{CAlg}$ is connective, i.e. $H^i(A) = 0$ for $i \geq 1$; so all the results in this section will be stated in this special situation. The reader interested in the more general non-connective case should consult [32, Chapter 7].

5.2.1 Flat and projective modules

Flat modules

Definition 5.2.2. A module $M \in \mathrm{Mod}_R$ is *flat* if

(a) $H^0(M)$ is a flat $H^(0)(R)$-module;

(b) for all $k \in \mathbb{Z}$ the canonical map

$$H^k(R) \otimes_{H^0(R)} H^0(M) \xrightarrow{\simeq} H^k(M)$$

is an isomorphism.

Remark 5.2.3. In [56] condition (b) in Definition 5.2.2 is called *strong*.

We notice that, by the standing assumption that R is connective, any flat R-module M is connective, i.e. $M \in \mathrm{Mod}_R^{\leq 0}$. Moreover, if R is discrete, then $M \simeq H^0(M)$ and $M \in \mathrm{Mod}_R^{\heartsuit}$.

The following result characterizes flatness in terms of properties of the tensor product

Proposition 5.2.4. *Given a module $M \in Mod_R^{\leq 0}$ then the following are equivalent*

(i) M is flat;

(ii) for every $N \in Mod^{\geq 0}$ one has $N \otimes_A M \in Mod_R^{\geq 0}$;

(ii') for every $N \in Mod^{\heartsuit}$ one has $N \otimes_A M \in Mod_R^{\heartsuit}$;

(iii) M is a filtered colimit of projective R-modules (see Definition 5.2.11 below);

(iii)' M is a filtered colimit of finitely generated free R-modules.

We first will need the following result which shows that flat R-modules are a useful computational tool.

Lemma 5.2.5. *Consider $M \in Mod_R^{\mathrm{flat}}$ a flat R-module and an arbitrary $N \in Mod_R$, then*

$$H^k(M \otimes_R N) \simeq H^0(M) \otimes_{H^0(R)} H^k(N)$$

for all $k \in \mathbb{Z}$.

Proof. Apply a spectral sequence to compute the left-hand side. Figure out a way to quickly explain this. ☐

Proof of Proposition 5.2.4. The implication (i) ⇒ (ii)/(ii)' follows directly from Lemma 5.2.5. The equivalence between (ii) and (ii)' we leave as an exercise for the reader. For (ii)' ⇒ (i) we notice that it is clear that $H^0(M)$ is flat, indeed for any discrete object $N \in \mathrm{Mod}_R$ one has Get the right formula here!

$$H^k(M \otimes_R N) \simeq \oplus_{i=0}^k \mathrm{Tor}^i_{H^0(R)}(H^{k-i}(M), H^0(N)).$$

The above formula also implies that condition (b) is satisfied for $k \geq 1$. For $k \leq -1$ one proceeds by induction. The direction (i) ⇒ (iii)' is trickier, this is proved in [32]*7.2.2.15. (iii') ⇒ (iii) is tautological.

Finally for (iii) $\Rightarrow$ (ii)' we first argue that any projective R-module M is flat. It is clear that any free R-module is flat. Now let

$$\begin{array}{ccccc} & & F & & \\ & \overset{i}{\nearrow} & & \overset{r}{\searrow} & \\ M & & \xrightarrow{\mathrm{id}_M} & & M \end{array}$$

be a diagram expressing M as a retract of a free R-module F. By [33]*4.4.5.18 one obtains that

$$M \simeq \operatorname{colim}\left(F \xrightarrow{e} F \xrightarrow{e} \cdots\right)$$

where $e := \mathrm{id}_F - i \circ r$. Let N be a discrete R-module, since $(-) \otimes_R N$ commutes with filtered colimits one has

$$M \otimes_R N \simeq \operatorname{colim}\left(N \otimes_R F \xrightarrow{e} N \otimes_R F \xrightarrow{e} \cdots\right).$$

Since F is flat, each term is discrete and hence $M \otimes_R N$ is discrete since the filtered colimit of discrete objects is discrete. Indeed, the colimit can be computed in the ∞-category of spaces, where it is given by a homotopy colimit, but since it is filtered it is just an ordinary colimit (Reference for this!?). Then one checks directly that $\pi_i(-)$ commutes with filtered colimits. The same argument shows that the filtered colimit of flat R-modules is flat, so we are done. □

Condition (ii) has the following natural generalization which will be useful later when trying to understand the relation between almost perfect objects and perfect objects in Mod_R.

Definition 5.2.6.
- Let $n \geq 0$, a module $M \in \mathrm{Mod}_R$ is said to have *Tor-amplitude* $\leq n$ if the functor $(-) \otimes_R M$ restricted to discrete objects factors as follows:

$$(-) \otimes_R M : \mathrm{Mod}_R^{\heartsuit} \to \mathrm{Mod}_R^{\geq -n}.$$

- Given two integers $a \leq b$ we say that $M \in \mathrm{Mod}_R$ has *Tor amplitude in [a,b]* if for very discrete module $N \in Mod_R^{\heartsuit}$ one has

$$N \otimes_R M \in \mathrm{Mod}_R^{[-b,-a]}.$$

Remark 5.2.7. From the definition we have that if M has Tor-amplitude $\leq n$ then M has Tor-amplitude in $[a, n]$ for every $a \leq n$, i.e.by abuse of notation, we could say that M has Tor-amplitude in $(-\infty, n]$. Conversely, if M has Tor-amplitude in $[a, b]$ then M has Tor-amplitude $\leq b$.

Remark 5.2.8. Since for a connective derived ring R one has that $(-)\otimes_R$ Recall that if $M \in \mathrm{Mod}_R^{\leq 0}$ one has that $M \otimes_R (-)$ preserves connective objects, since R is connective. This imply that any module $M \in \mathrm{Mod}_R^{\leq n}$ has Tor-amplitude in $[-n, \infty)$, i.e. for every $b \geq n$ M has Tor-amplitude in $[-n, b]$.

Remarks 5.2.7 and 5.2.8 together give the following

Lemma 5.2.9. *Given a module $M \in Mod_R$ and two integers $a \leq 0 \leq b$ the following are equivalent:*

(i) M has Tor-amplitude in $[a, b]$;

(ii) M is $-a$ connective, i.e. $M \in Mod^{\leq -a}$, and M has Tor-amplitude $\leq b$.

Mention all the consequences of Tor amplitude from [2, Proposition 2.13]. The following collects many useful properties about Tor-amplitude

Lemma 5.2.10. *(i) If M has Tor-amplitude in $[a, b]$, then any retract of M has Tor-amplitude in $[a, b]$.*

(ii)

Proof. For (i) let

$$\begin{array}{ccccc} & & F & & \\ & \overset{i}{\nearrow} & & \overset{r}{\searrow} & \\ M & & \xrightarrow{\mathrm{id}_M} & & M \end{array}$$

be a diagram exhibiting M as a retract of an R-module F of Tor amplitude in $[a, b]$. Then given any discrete module N one obtains a diagram

$$\begin{array}{ccccc} & & F & & \\ & \overset{i\otimes\mathrm{id}_N}{\nearrow} & & \overset{r\otimes\mathrm{id}_N}{\searrow} & \\ M\otimes N & & \xrightarrow{\mathrm{id}_{M\otimes N}} & & M\otimes N \end{array}$$

which induces a diagram on cohomology

$$\begin{array}{ccccc} & & H^i(F) & & \\ & \nearrow & & \searrow & \\ H^i(M\otimes N) & & \xrightarrow{\mathrm{id}_{H^i(M\otimes N)}} & & H^i(M\otimes N) \end{array}$$

for every $i \in \mathbb{Z}$. From this it is clear that $H^i(M \otimes N)$ vanishes for $i \notin [a, b]$.

(...) □

Projective modules There is also a very well-behaved theory of projective R-modules for R a derived ring that encompasses analogues of the usual properties.

Definition 5.2.11. An R-module $M \in \mathrm{Mod}_R$ is said to be *projective* if

(a) M is connective;

(b) M is a projective object of $\mathrm{Mod}_R^{\leq 0}$, i.e. $\mathrm{Hom}_R(M, -)$ commutes with geometric realizations.

In particular, since R is assumed connective one sees that R is projective. Moreover, in condition (b) it is essential that we consider the object co-represented by M in $\mathrm{Mod}_R^{\leq 0}$ and *not* in Mod_R, since the latter doesn't have any non-trivial projective objects.

The following result shows how many usual properties from the homological of projective modules generalize to the commutative algebra of derived rings.

Proposition 5.2.12. *For a connective module $M \in Mod_R^{\leq 0}$ the following are equivalent:*

(i) M is projective;

(ii) for all $N \in Mod_R^{\leq 0}$

$$Ext^1_R(M, N) = 0;$$

(iii) for all $N \in Mod_R^{\leq 0}$

$$Ext^k_R(M, N) = 0$$

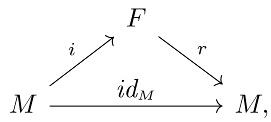
for all $k \geq 1$;

(iii)' for all $N \in Mod^{\heartsuit}$

$$Ext^k_R(M, N) = 0$$

for all $k \geq 1$;

(iv) M is a retract of a free R-module, i.e. there exists a commutative diagram in Mod_R



where F is a free R-module;

(v) given any fiber sequence

$$N' \to N \to N''$$

in $\mathrm{Mod}_R^{\leq 0}$ the induced morphism

$$Hom_{\mathrm{h}\,Mod_R}(M,N) \to Hom_{\mathrm{h}\,Mod_R}(M,N'')$$

is surjective;

(vi) M is flat and $H^0(M)$ is a projective $H^0(R)$-module.

Proof. For (i) $\Rightarrow$ (ii) we recall that by definition one wants to calculate

$$\mathrm{Ext}^k_R(M,N) := \mathrm{Hom}_{\mathrm{h}\,\mathrm{Mod}_R}(M[-k],N) \simeq \mathrm{Hom}_{\mathrm{h}\,\mathrm{Mod}_R}(M,N[k]).$$

For $k=1$, let $P_n := 0 \underset{N[1]}{\times} \underset{N[1]}{\times} 0 \underset{N[1]}{\times} \cdots \underset{N[1]}{\times} 0$, where we include $(n+1)$-factors of $N[1]$. This gives a simplicial object in $\mathrm{Mod}_R^{\leq 0}$, whose geometric realization by definition recovers N, i.e. $|P_\bullet| \simeq N$. We are interested in $\pi_0\left(\mathrm{Hom}_{\mathrm{Mod}_R}(M,|P_\bullet|)\right)$, but for geometric realization of spaces (Give a reference for this.) one has

$$\pi_0\left(\mathrm{Hom}_{\mathrm{Mod}_R}(M,|P_\bullet|)\right) \simeq \mathrm{Coker}\left(\pi_0(\mathrm{Hom}_{\mathrm{Mod}_R}(M, 0 \underset{Q[1]}{\times} 0)) \to \pi_0(\mathrm{Hom}_{\mathrm{Mod}_R}(M,0))\right).$$

Since 0 is a final object, one has $\pi_0(\mathrm{Hom}_{\mathrm{Mod}_R}(M,0)) \simeq 0$, so $\mathrm{Ext}^1_R(M,N) \simeq 0$.

(iii) $\Rightarrow$ (ii) is tautological and (ii) $\Rightarrow$ (iii) by noticing that

$$\mathrm{Ext}^1(M,N[i-1]) \simeq \mathrm{Ext}^i(M,N)$$

for any $i \geq 1$.

(iii) $\Rightarrow$ (iii)' is also tautological.

For (iii)' $\Rightarrow$ (iii) one needs the fact that the t-structure of Mod_R is left-complete, i.e. $\mathrm{Mod}_R \simeq \lim_{n\geq 1}\mathrm{Mod}_R^{\geq -n}$. Given $N \in \mathrm{Mod}_R^{\leq 0}$ for any $i \geq 0$ one has a fiber/cofiber sequence

$$H^{-i}(N)[i] \to \tau^{\geq -i}N \to \tau^{\geq -i+1}N$$

which induces an exact sequence of abelian groups

$$\mathrm{Ext}^{k+i}_{\mathrm{Mod}_R}(M,H^{-i}(N)) \to \mathrm{Ext}^k_{\mathrm{Mod}_R}(M,\tau^{\geq -i}(N)) \to \mathrm{Ext}^k_{\mathrm{Mod}_R}(M,\tau^{\geq -i+1}(N)) \to \mathrm{Ext}^{k+i+1}_{\mathrm{Mod}_R}(M,H^{-i}(N)).$$

Condition (iii)' implies that the tower of abelian groups $\{\mathrm{Ext}^k_{\mathrm{Mod}_R}(M,\tau^{\geq -i}(N))\}_{k\geq 0}$ is stabilizes for $k > -i$. By left-completeness of Mod_R one has $N \simeq \lim_{i\geq 0}\tau^{\geq -i}Q$ so that for any $k > -i$ one has

$$\mathrm{Ext}^k(M,N) \simeq \mathrm{Ext}^k(M,\tau^{\geq -i}N).$$

In particular, for $i > 0$ one has $\mathrm{Ext}^k(M,N) \simeq \mathrm{Ext}^k(M,H^0(N)) = 0$.

(iv) $\Rightarrow$ (vi) that any free R-module is flat is clear and that retracts of flat are flat follow from Lemma 5.2.10 (i). Consider $M \in \mathrm{Mod}_R^{\leq 0}$ and $N \in \mathrm{Mod}_R^{\heartsuit}$ then one has a cofiber/fiber sequence

$$\mathrm{Hom}_{\mathrm{Mod}_R}(H^0(M),N) \to \mathrm{Hom}_{\mathrm{Mod}_R}(M,N) \to \mathrm{Hom}_{\mathrm{Mod}_R}(\tau^{\leq -1}(M),N)$$

which induces a long exact sequence

$$\begin{array}{ccccc}
 & & \cdots & \longrightarrow & H^{-1}(\mathrm{Hom}_{\mathrm{Mod}_R}(\tau^{\leq -1}(M), N)) \\
H^0(\mathrm{Hom}_{\mathrm{Mod}_R}(H^0(M), N)) & \longrightarrow & H^0(\mathrm{Hom}_{\mathrm{Mod}_R}(M, N)) & \longrightarrow & H^0(\mathrm{Hom}_{\mathrm{Mod}_R}(\tau^{\leq -1}(M), N)) \\
H^1(\mathrm{Hom}_{\mathrm{Mod}_R}(H^0(M), N)) & \longrightarrow & H^1(\mathrm{Hom}_{\mathrm{Mod}_R}(M, N)) & \longrightarrow & H^1(\mathrm{Hom}_{\mathrm{Mod}_R}(\tau^{\leq -1}(M), N)) \\
H^2(\mathrm{Hom}_{\mathrm{Mod}_R}(H^0(M), N)) & \longrightarrow & \cdots & & .
\end{array}$$

Since $H^i(\mathrm{Hom}_{\mathrm{Mod}_R}(\tau^{\leq -1}(M), N)) = 0$ for $i \leq 0$, this implies that

$$H^1(\mathrm{Hom}_{\mathrm{Mod}_R}(H^0(M), N)) \simeq \mathrm{Ext}^1_{\mathrm{Mod}_R}(H^0(M), N) \hookrightarrow \mathrm{Ext}^1_{\mathrm{Mod}_R}(M, N) \simeq H^1(\mathrm{Hom}_{\mathrm{Mod}_R}(M, N))$$

is injective. Thus, one obtains that $\mathrm{Ext}^1(H^0(M), N)$ vanishes for any discrete N, that is $H^0(M)$ is projective.

(vi) $\Rightarrow$ (i). We first consider the case where $H^0(M)$ is a finitely generated free $H^0(R)$-module, then by picking lifts of the generators $\{m_i\}_I$ of $H^0(M)$ one has a morphism

$$\oplus_I R \to M$$

which induces an isomorphism on H^0. Since M is flat this implies that M is a free R-module so in particular projective.

A general projective module $H^0(M)$ can be realized as a direct summand $N_0 \oplus H^0(M) \simeq F_0$ where F_0 is a free $H^0(R)$-module. Moreover, we claim that we can suppose that N_0 free. Indeed, if that is not the case, let $M' := M \oplus H^\bullet(R) \otimes_{H^0(R)} N_0$

let $N_0 \oplus N_1 \simeq F_1$ where N_1 is a projective $H^0(R)$-module and F_1 is free. One considers

$$\oplus_{\mathbb{N}}(F_1 \oplus H^0(M)) \simeq (...)$$

Understand and write Proposition 7.2.2.18. □

Remark 5.2.13. Notice that condition (iv) in Proposition 5.2.12 corresponds to (possibly) the most straightforward generalization from the classical notion of a projective R-module and is easily seen to recover the usual notion when restricted to a discrete R-module.

Injective modules Write a discussion of injective modules following [50].

5.2.2 Perfect and almost perfect modules

There are two important finiteness conditions in the context of modules over a derived ring R. We first formulate the more general one, that of almost perfect modules, though which is harder to have an intuition for. And then we formulate the stronger condition–perfect modules–which can be described in three different ways.

Definition 5.2.14. An R-module M is said to be *almost perfect* if

(a) $M \in \mathrm{Mod}_R^{\leq k}$ for some $k \in \mathbb{Z}$;

(b) for every $n \geq 0$ the object $\tau^{\geq -n}(M)$ is compact in $\mathrm{Mod}_R^{\geq -n}$.

We will let APerf_R denote the full subcategory of Mod_R generated by almost perfect R-modules.

Example 5.2.15. For any derived ring R, one clearly has that R is almost perfect.

The condition of being almost perfect can be more easily described in the case in which R satisfies some finiteness itself. For that we need a definition:

Definition 5.2.16. A derived ring $R \in \mathrm{CAlg}$ is said to be *Noetherian* if it satisfies:

(i) $H^0(R)$ is Noetherian as an ordinary ring;

(ii) $H^i(R)$ is a finitely generated (equivalently presented) $H^0(R)$-modules, for each $i \in \mathbb{Z}$.

Proposition 5.2.17. *For R a Noetherian derived ring, an R-module M is almost perfect if and only if*

(i) $H^i(M) = 0$ for $i \gg 0$;

(ii) $H^i(M)$ is a finitely generated $H^0(R)$-module, for every $i \in \mathbb{Z}$.

Proof. Condition (i) and condition (a) in Definition 5.2.14 are exactly the same.

Notice that since M is almost perfect if and only if a shift of M is almost perfect, so it is enough to consider the case where M is connective. Assume that M is connective and almost perfect, we will prove by induction on i that $H^{-i}(M)$ is finitely generated. The base case follows from the fact that $H^0(M)$ is a compact object in $\mathrm{Mod}_R^{\geq 0}$. Indeed, we notice that the inclusion $\imath^{\geq 0} : \mathrm{Mod}_R^{\heartsuit} \to \mathrm{Mod}_R^{\geq 0}$ is a fully faithful left adjoint, with right adjoint $\tau^{\geq 0} : \mathrm{Mod}_R^{\geq 0} \to \mathrm{Mod}_R^{\heartsuit}$. Hence given a filtered diagram $\{N_i\}$ of objects in $\mathrm{Mod}_R^{\heartsuit}$ one has

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}_R^{\heartsuit}}(H^0(M), \mathrm{colim}_I N_i) &\simeq \mathrm{Hom}_{\mathrm{Mod}_R^{\geq 0}}(\imath^{\geq 0} \circ H^0(M), \imath^{\geq 0}(\mathrm{colim}_I N_i)) \\
&\simeq \mathrm{Hom}(\imath^{\geq 0} \circ \tau^{', \geq 0}(M), \mathrm{colim}_I \imath^{\geq 0}(N_i)) \\
&\simeq \mathrm{colim}_I \mathrm{Hom}(\imath^{\geq 0} \circ \tau^{', \geq 0}(M), \imath^{\geq 0}(N_i)) \\
&\simeq \mathrm{colim}_I \mathrm{Hom}(\tau^{' \geq 0}(M), \tau^{' \geq 0} \circ \imath^{\geq 0}(N_i)) \\
&\simeq \mathrm{colim}_I \mathrm{Hom}(H^0(M), N_i)
\end{aligned}$$

where in the second and last isomorphisms we used that $H^0 M \simeq \tau^{' \geq 0} M$ and in the third we used the condition that M is connective to identify $\tau^{' \geq 0} \circ \tau^{\leq 0}(M)$ with $\tau^{\geq 0}(M)$ which is compact. Because $\mathrm{Mod}_R^{\heartsuit}$ is equivalent to the ordinary category of $H^0(R)$-modules the result follows from the fact that an $H^0(R)$-module is a compact object if and only it is finitely presented (see [Stacks, Tag 0G8P]).

For the inductive case, consider a finitely generated and free R-module P with a map $\alpha : P \to M$ such that $H^0(\alpha)$ is surjective. Then one has that $K := \mathrm{Fib}(\alpha)$ is connective and by the inductive hypothesis $H^i(K)$ is finitely generated for $i > -n$. Then from the long exact sequence in cohomology we obtain that $H^{-n}(M)$ fits into a short exact sequence

$$0 \to \mathrm{Coker}(H^{-n}(K) \to H^{-n}(P)) \to H^{-n}(M) \to \mathrm{Ker}(H^{-n+1}(K) \to H^{-n+1}(P)) \to 0. \tag{5.4}$$

Since P is a finitely generated R-module and R is Noetherian, each $H^i(P)$ is finitely generated and one can easily directly check using usual commutative algebra that the left and right term in (5.4) are finitely generated.

Conversely, assume that M is connective and satisfies conditions (i) and (ii). We claim that we can recover M as the colimit of a diagram

$$0 \xrightarrow{f_0} D(0) \xrightarrow{f_1} D(1) \xrightarrow{f_2} \to \cdots$$

where for all $n \geq 0$ each $D(n)$ is almost perfect and each $\mathrm{Cofib}(f_n)[-n]$ is a finitely generated free R-module. We construct such diagram inductively. The base case is given by considering $g_0 : P_0 \to M$ a map from a finitely generated free R-module P_0 such that $H^0(g_0)$ is surjective. Let $D(0) := P_0$ and $\alpha_0 := g_0 : D(0) \to M$. Now let $K_0 := \mathrm{Fib}(\alpha_0)$, since $H^0(K_0)$ is finitely generated consider $\alpha_1 : P_1 \to K_0$ where P_1 is a finitely generated free R-module. Then one has the composite

$$g_1 : P_1 \to K_0 \to D(0)$$

so we let $D(1) := \mathrm{Cofib}(g_1)$. Notice that since α_0 restricted to P_1 is homotopically trivial, one has an induced map $\alpha_1 : D(1) \to M$. Let

$$D(0) \stackrel{f_0}{\to} D(1) \stackrel{\alpha_1}{\to} M$$

where the composite is α_0. By the octahedral axiom of a stable ∞-category one obtains the following cofiber/fiber sequence

$$\mathrm{Cofib}(f_0) \to \mathrm{Cofib}(\alpha_0) \to \mathrm{Cofib}(\alpha_1)$$

where $\mathrm{Cofib}(f_0) \simeq P_1[1]$. Hence the long exact sequence in cohomology gives that $\mathrm{Fib}(\alpha_1)$ belongs to $\mathrm{Mod}_R^{\leq -1}$. Finally, we need to check that $H^{-1}(\alpha_1)$ is a finitely generated $H^0(R)$-module. Since $D(1)$ is almost perfect, by the implication already proved one has that $H^i(D(1))$ is finitely generated for all $i \in \mathbb{Z}$. Now again the short exact sequence

$$0 \to \mathrm{Coker}(H^{-2}(D(1)) \to H^{-2}(M)) \to H^{-1}(\mathrm{Fib}(\alpha_1)) \to \mathrm{Ker}(H^{-1}(D(1)) \to H^{-1}(M)) \to 0$$

gives that $H^{-1}(\mathrm{Cofib}(\alpha_1))$ is finitely generated. The nth step is proved exactly in the same way, so we leave it as an exercise. □

Definition 5.2.18. An R-module M is said to be *perfect* if it is a perfect object of Mod_R, i.e. the functor $\mathrm{Hom}_{\mathrm{Mod}_R}(M, -) : \mathrm{Mod}_R \to \mathrm{Spc}$ commutes with filtered colimits. We let Perf_R generate the full subcategory of Mod_R generated by perfect R-modules.

Remark 5.2.19. Notice that since conditions (a) and (b) in Definition 5.2.14 are preserved under finite colimits and shifts, any perfect R-module is almost perfect, i.e. one has a fully faithful embedding

$$\mathrm{Perf}_R \hookrightarrow \mathrm{APerf}_R.$$

Before giving a list of equivalent characterization of perfect modules we mention on useful fact in interpreting the category Mod_R.

Lemma 5.2.20 ([32]*Proposition 7.2.4.3)**.** *One has an equivalence of categories*

$$\begin{aligned} Mod_R &\stackrel{\simeq}{\to} Fun^{\mathrm{w-cont.}}(Mod_A, Spctr), \\ M &\mapsto M \otimes_R (-) \end{aligned}$$

where the superscript w−cont. in Fun means we consider only weakly continuous functor, i.e. those preserving filtered colimits.

Proposition 5.2.21. *Let $M \in Mod_R$ the following are equivalent:*

(i) M is a perfect R-module;

(ii) M is a retract of $\oplus_I R[n_i]$ for some finite set I;

(iii) M is dualizable, i.e. there exists $M^\vee$ and maps u and c such that

$$c \otimes id_M \circ id_M \otimes u \simeq id_M \quad and \quad id_{M^\vee} \otimes c \circ u \otimes id_{M^\vee} \simeq id_{M^\vee};$$

(iv) M is almost perfect and has finite Tor amplitude.

Proof. (ii) ⇒ (i). R is clearly compact and we claim that the retract of any compact object is also compact. Indeed, let F be a direct sum of shifts of free R-modules such that M is a retract of F. Then the diagram

$$\begin{array}{ccc} & \mathrm{Hom}_{\mathrm{Mod}_R}(F, -) & \\ \stackrel{f}{\nearrow} & & \stackrel{r}{\searrow} \\ \mathrm{Hom}_{\mathrm{Mod}_R}(M, -) & \xrightarrow{\quad \mathrm{id} \quad} & \mathrm{Hom}_{\mathrm{Mod}_R}(M, -) \end{array} \tag{5.5}$$

of objects in $\mathrm{Fun}(\mathrm{Mod}_R, \mathrm{Spc})$ exhibits $\mathrm{Hom}_{\mathrm{Mod}_R}(M,-)$ as a retract of $\mathrm{Hom}_{\mathrm{Mod}_R}(F,-)$ in $\mathrm{Fun}(\mathrm{Mod}_R, \mathrm{Spc})$, or equivalently, as a retract in $\mathrm{h}\,\mathrm{Fun}(\mathrm{Mod}_R, \mathrm{Spc})$. For every object $X \in \mathrm{Mod}_R$ we let r_X and f_X denote the corresponding maps of the diagram (**??**) evaluated at X. For a filtered diagram $\{X_i\}_I$ we notice that the composite

$$\mathrm{Hom}(M, \mathrm{colim}_I X_i) \overset{f_{\mathrm{colim}_I X_i}}{\to} \mathrm{Hom}(F, \mathrm{colim}_I X_i) \overset{\simeq}{\leftarrow} \mathrm{colim}_I \mathrm{Hom}(F, X_i) \overset{\mathrm{colim}_I r_{X_i}}{\to} \mathrm{colim}_I \mathrm{Hom}(M, X_i)$$

is a homotopy inverse to the canonical map $\mathrm{colim}_I \mathrm{Hom}(M, X_i) \to \mathrm{Hom}(M, \mathrm{colim}_I X_i)$, hence M is compact.

(i) $\Rightarrow$ (ii) Here is a little intuition for an argument. Let let $N_+ \in \mathbb{Z}$ be the largest integer such that $H^N(M) \neq 0$. One can find a finitely generated free R-module P_{N_+} and a map $\alpha_{N_+} : P_{N_+} \to M$ such that $H^{N_+}(\alpha_{N_+})$ is surjective. We claim that after repeating this process finitely many times one obtains a direct of finitely generated free R-modules $\alpha : \oplus_{N_- \leq i \leq N_+} P_i$ and a map $\alpha : F \to M$ such that for all $i \in \mathbb{Z}$ $H^i(\alpha)$ is surjective. Indeed, finish this argument. I don't quite know how!

For (i) $\Rightarrow$ (iii) we notice that if M is compact the functor $\mathrm{Hom}_{\mathrm{Mod}_R}(M,-)$ preserves filtered colimits, so it follows from Lemma 5.2.20 that there exists $M^\vee \in \mathrm{Mod}_R$ such that

$$M^\vee \otimes (-) \simeq \mathrm{Hom}_{\mathrm{Mod}_R}(M,-). \tag{5.6}$$

Now we notice that the data of an isomorphism of functors as in (**??**) is equivalent to the data of a evaluation and coevaluation map exhibiting $M^\vee$ as the dual of M Maybe give a reference for this!.

For (i) $\Rightarrow$ (iv) we leave as an exercise to the reader to check that if $M \in \mathrm{Mod}_R$ is compact then for any $n \in \mathbb{Z}$ $\tau^{\geq n}(M)$ is compact in $\mathrm{Mod}_R^{\geq n}$. By the equivalence between (i) and (ii) already proved the claim about finite Tor-amplitude follows from the fact that any R-module of the form $\oplus_I R[n_i]$ with I finite has finite Tor amplitude and that retracts of a module of finite Tor amplitude still have finite Tor amplitude.

Finally, for (iv) $\Rightarrow$ (i) if M has finite Tor amplitude then there exists $n \in \mathbb{Z}$ such that $M[n]$ is flat. Thus, it is enough to prove the result in the case where M is connective, in which case we can assume that M has Tor amplitude $\leq m$ for some $m \geq 0$. The case when $m = 0$ follows from Proposition **??**. ☐

Warning 5.2.22. Notice that the usual 1-categorical proof of the fact that retracts of compact objects are compact doesn't apply in the ∞-categorical context, since if an object M is a retract of a compact object F in an ∞-category $\mathscr{C}$ then one only can recover M as

$$M \simeq \mathrm{colim}_{n \geq 1} F^{(f \circ r)^n}$$

where in general this is not a finite colimit, so one when considering the functors co-represented by these evaluated at a colimit diagram one can *not* commute the associated (not necessarily) limit diagram past a filtered colimit.

Remark 5.2.23. In particular, if R is discrete one has an equivalence

$$\mathrm{APerf}_{H^0(R)} \simeq \{M \in \mathrm{Mod}^-_{H^0(R)} \mid M \overset{\mathrm{q.-iso.}}{\simeq} [0 \to P_{-m} \to \cdots \to P_n \to 0],\ P_i \text{ is f.g. projective} \forall i \in \mathbb{Z}\}.$$

Indeed, we will construct the representative $[0 \to P_{-m} \to \cdots \to P_n \to 0]$ by induction. Up to a shift we can suppose that $M \in \mathrm{Mod}^{\leq 0}_{H^0(R)}$. Then, one obtains $\tau^{\geq 0}(M) \simeq H^0(M)$ is finitely presented and flat hence it is projective. Let $\alpha : P_0 \to M$ be such that $H^0(P) \overset{\simeq}{\to} H^0(M)$, then one can see that

$$\mathrm{Fib}(\alpha) \in \mathrm{Mod}^{\leq -1}_{H^0(R)}.$$

By Proposition 5.2.21 (iv) one has that M has finite Tor-amplitude, say $[-n, 0]$ for some $n \geq 0$, moreover one easily check that $\mathrm{Fib}(\alpha)$ has Tor-amplitude in $[-n+1, 0]$. Thus, by induction we obtain a finite resolution by finitely generated projective objects in $\mathrm{Mod}_{H^0(R)}$ which are equivalent to usual f.g. projective modules, i.e. they are concentrated in the heart of the natural t-structure.

5.2.3 Vector bundles

In this section we consider the intersection of the homological conditions from §5.2.1 and the finiteness conditions from 5.2.2. We start with the following observation:

Definition-Proposition 5.2.24. An R-module M is said to be a *vector bundle*, or a *finitely generated projective* R-module, if it satisfies any of the following equivalent conditions:

(i) M is a compact and projective of $\mathrm{Mod}_R^{\leq 0}$;

(ii) M is a retract of a finitely generated free R-module;

(iii) M is almost perfect and flat;

(iv) M is dualizable in the category of connective R-modules.

Proof. (i) $\Rightarrow$ (ii) by Proposition 5.2.21 M compact implies that M is a retract of $\oplus_I R[n_i]$ for a finite set I and $n_i \in \mathbb{Z}$, since it is also flat one has that $n_i = 0$ for all $i \in I$.

(ii) $\Rightarrow$ (i) follows from Proposition 5.2.21 and Proposition 5.2.12.

(ii) $\Rightarrow$ (iii) is clear: Proposition 5.2.4 gives flatness and Proposition 5.2.21 and Remark 5.2.19 gives almost perfectness.

For (iii) $\Rightarrow$ (ii) if M is flat and almost perfect we have that $H^0(M)$ is a flat and finitely presented $H^0(R)$-module by [Stacks, Tag 00NX] $H^0(M)$ is projective. Now Proposition 5.2.12 gives that M is projective and its proof that M is actually finitely generated.

For (i) $\Rightarrow$ (iv) follows from the collection of dualizable objects in $\mathrm{Mod}_R^{\leq 0}$ being closed under finite direct sums and taking summands. Indeed, Give a more detailed argument.

Finally (iv) $\Rightarrow$ (iii) first we notice that M is also dualizable in Mod_R, so by Remark 5.2.19 it is almost perfect. Now we notice that for any $N \in \mathrm{Mod}_R^{\geq 0}$ one has

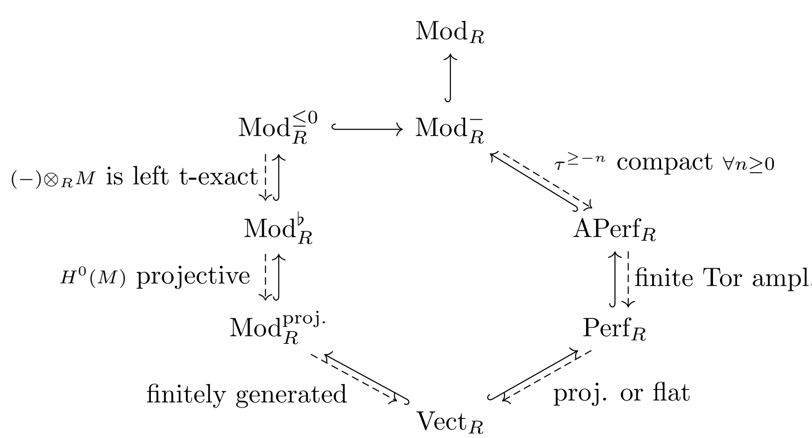
$$H^i(N \otimes_R N) \simeq H^i(\mathrm{Hom}_{\mathrm{Mod}_R}(M^\vee, N)) \simeq \mathrm{Hom}_{h\mathrm{Mod}_R}(M^\vee[-i], N) = 0,$$

for all $i \leq 1$ since $M^\vee \in \mathrm{Mod}_R^{\leq 0}$, which gives that M is flat. □

Remark 5.2.25. When R is classical, i.e. $R \xrightarrow{\simeq} H^0(R)$, one notices that (iii) implies that any object $M \in \mathrm{Vect}_{H^0(R)}$ is discrete, i.e. $H^k(M) = 0$ for all $k \neq 0$. In particular, the condition of being almost perfect implies that $\tau^{\geq 0} \circ \tau^{\leq 0}(M) \simeq H^0(M)$ is a compact object of $\mathrm{Mod}_{H^0(R)}^{\heartsuit}$, that it is finitely presented. Since $H^0(M)$ is also flat, one has that $H^0(M)$ is finitely generated and projective, hence a vector bundle in the usual sense.

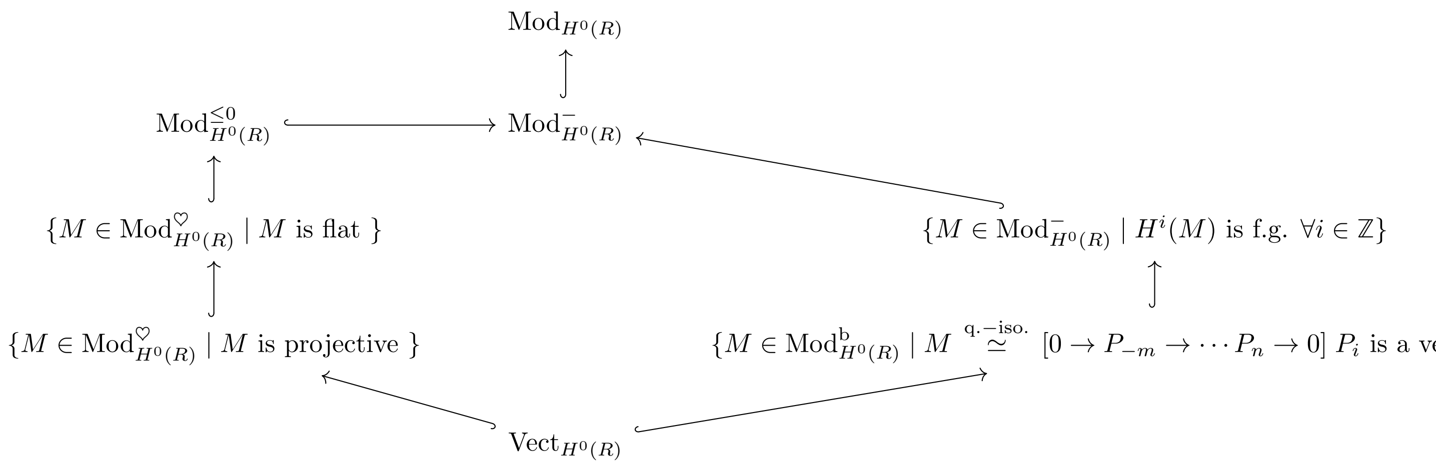
5.2.4 Summary

We collect here all the results concerning the different conditions that one can impose on R-modules for R a derived ring.



(5.7)

For the underlying (classical) ring $H^0(R)$ the diagram above specializes to



(5.8)

Exercise 5.2.26. Let k be a discrete field, describe all the categories in the above diagram for

(a) $R = k$;

(b) $R = k[\epsilon]$ where $|\epsilon| = -1$;

(c) $R = k[\eta]$ where $|\eta| = -2$.

5.3 A strict model presentation

For k a field, let $\mathsf{Ch}(k)$ be the monoidal model category of chain complexes of k-vector spaces Explain what monoidal model category is.. Then there exits a pair of adjoint functors

$$\mathsf{CAlg}(\mathsf{Ch}(k)) \underset{\text{oblv}}{\overset{\text{Sym}(-)}{\leftrightarrows}} \mathsf{Ch}(k)$$

which endow $\mathsf{CAlg}(\mathsf{Ch}(k))$ with the unique model structure such that these functors form a Quillen adjunction Insert a reference here explaining how this works..

Assume that $\mathbb{Q} \subseteq k$. Then [32]*Proposition 4.5.4.7 implies that

$$\mathrm{N}\left(\mathsf{CAlg}(\mathsf{Ch}(k))_{\mathrm{c}}\right)[W'^{-1}] \overset{\simeq}{\to} \mathrm{CAlg}\left(\mathrm{N}\left(\mathsf{Ch}(k)_{\mathrm{c}}\right)[W^{-1}]\right),$$

where the left-hand side denotes the ∞-category underlying the monoidal model category whose objects are *cofibrant* Check this! commutative differential graded k-algebras. Notice that the right-hand side is equivalent to $\mathrm{CAlg}(\mathscr{D}(k))$ or to $\mathrm{CAlg}(\mathrm{Mod}_k)$ by Proposition 5.1.18 and fill reference with place where we compare the tensor structure.

5.3.1 The left module ∞-operad

Chapter 6

Prestacks and schemes

In this chapter we introduce prestacks, topologies on affine schemes and stacks. Once the hard work of constructing the ∞-categories of derived rings with all of its properties and having a simple way to describe presheaves and sheaves in the ∞-categorical language it is reasonably straight-forward how to proceed.

6.1 Prestacks

6.1.1 Definition

We start by defining the category of affine schemes.

Definition 6.1.1. The category of *affine schemes* is the opposite of the category of derived rings

$$\mathrm{Sch}^{\mathrm{aff}} := (\mathrm{CAlg})^{\mathrm{op}}.$$

Remark 6.1.2. Note that we keep the convention that we will not specific what type of affine schemes we consider, since for the formalism of this section all three cases discussed in §4.1 can be treated simultaneously. It is important to notice that for *non-connective* $\mathbb{E}_\infty$-algebras one needs to make certain modifications in the formalism. This is the point of notion called *homotopical algebraic geometry context* in [56, §1.3.2].

Definition 6.1.3. A *prestack* is a functor

$$\mathscr{X} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

i.e. a presheaf of spaces on affine schemes. We let $\mathrm{PreStk} := \mathrm{Fun}((\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathrm{Spc})$ denote the category of such objects.

Remark 6.1.4. The notion of a prestack is so general that it is essentially impossible to say something non-formal about an arbitrary prestack. It is however useful in the sense that if one can make sense of constructions at this level of generality one obtains these constructions for all other geometry objects of interest.

Example 6.1.5. Given a scheme X, for each non-empty finite set I let X^I denote the product of $|I|$ copies of X. For a surjective map $f : I \twoheadrightarrow J$ one has a diagonal map

$$\begin{aligned} X^J &\hookrightarrow X^I \\ x_j &\mapsto \delta^i(x_j) \text{ where } \delta^i(x_j) = x_j \ \forall i \in f^{-1}(j). \end{aligned}$$

The *Ran space of* X is the colimit

$$\mathrm{Ran}(X) := \mathrm{colim}_{\mathrm{Fin}^{\mathrm{surj.}}} X^I$$

where $\mathrm{Fin}^{\mathrm{surj.}}$ denotes the category of non-empty finite sets with morphisms surjective maps. In general $\mathrm{Ran}(X)$ is only a prestack, in the sense that it doesn't satisfy descent (see below). It is however a very useful object in the geometric Langlands program [] or also in Gaitsgory–Lurie's proof of the Tamagawa conjecture [15].

6.1.2 Coconnectiveness

We let ${}^{\leq n}\mathrm{PreStk} := \mathrm{Fun}({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}, \mathrm{Spc})$ denote the category of functors from n-coconnective affine schemes to spaces, and

$$^{\leq n}(-) : \mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$$

the functor obtained by restriction along the inclusion ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$. The functor ${}^{\leq n}(-)$ admits a fully faithful[1] left adjoint formally given by left Kan extension

$$\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}.$$

Given an object $\mathscr{X}_0 \in {}^{\leq n}\mathrm{PreStk}$ we notice that informally the value of its left Kan extension to a prestack on an affine scheme S is given by

$$\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathscr{X}_0)(S) \simeq \underset{S \to S';|\ S' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}{\mathrm{colim}} \mathscr{X}_0(S').$$

Similarly to the case of affine schemes we will denote by

$$\tau^{\leq n} : \mathrm{PreStk} \overset{^{\leq n}(-)}{\to} {}^{\leq n}\mathrm{PreStk} \overset{\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}}{\to} \mathrm{PreStk}$$

the corresponding colocalization functor.

Definition 6.1.6. A prestack $\mathscr{X}$ is said to be *n-coconnective* if the canonical morphism

$$\tau^{\leq n}(\mathscr{X}) \to \mathscr{X}$$

is an isomorphism.

Remark 6.1.7. When $n = 0$ we will also use the following notations ${}^{c\ell}\mathrm{PreStk} := {}^{\leq 0}\mathrm{PreStk}$, ${}^{c\ell}(-) : \mathrm{PreStk} \to {}^{c\ell}\mathrm{PreStk}$, $\tau^{c\ell} := \tau^{\leq 0}$ and $\mathrm{LKE}_{c\ell} : {}^{c\ell}\mathrm{PreStk} \to \mathrm{PreStk}$.

Remark 6.1.8. Some references refer to a 0-coconnective prestack as a classical prestack. We will avoid making this abuse, since for us a *classical prestack* will mean an object of ${}^{c\ell}\mathrm{PreStk}$.

Example 6.1.9. Any $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ gives an n-coconnective prestack h_S via the Yoneda embedding, i.e.

$$h_S(-) := \mathrm{Maps}_{\mathrm{Sch}^{\mathrm{aff}}}(-, S) : \mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}.$$

The next condition is something that we expect on any prestack of a geometric nature, i.e. obtained as a well-behaved moduli space, or any derived scheme (see below for a definition).

For $S \in \mathrm{Sch}^{\mathrm{aff}}$ consider the functor

$$\mathbb{Z}_{\geq 0} \to \mathrm{Sch}^{\mathrm{aff}}_{/S}$$
$$n \mapsto \tau^{\leq n}(S).$$

Definition 6.1.10. A prestack $\mathscr{X}$ is *convergent*[2] if for any $S \in \mathrm{Sch}^{\mathrm{aff}}$ the canonical map

$$\mathscr{X}(S) \to \lim_{\mathbb{Z}_{\geq 0}} \mathscr{X}(\tau^{\leq n}(S))$$

is an isomorphism.

Exercise 6.1.11. (i) Let ${}^{\leq \infty}\mathrm{Sch}^{\mathrm{aff}}$ denote the subcategory of *eventually coconnective* affine scheme, i.e. $S \in {}^{\leq \infty}\mathrm{Sch}^{\mathrm{aff}}$ if $S \simeq \tau^{\leq n}(S)$ for some $n \in \mathbb{Z}_{\geq 0}$. Prove that $\mathscr{X}$ is convergent if and only if the canonical map

$$\mathscr{X} \to \mathrm{RKE}_{^{\leq \infty}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathscr{X}|_{^{\leq \infty}\mathrm{Sch}^{\mathrm{aff}}})$$

is an isomorphism.

[1] Indeed, since the unit

$$\mathrm{id}_{^{\leq n}\mathrm{PreStk}} \to {}^{\leq n}\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}$$

is an isomorphism.

[2] Lurie (cf. [35, Definition 17.3.2.1]) uses the term *nilcomplete* for what we call convergent.

(ii) Any affine scheme is convergent.

(iii) The prestack $\mathscr{Q}$ defined as the composite

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \overset{\mathrm{QCoh}_{(-)^*}}{\to} \mathrm{Cat}_\infty \overset{(-)^{\simeq}}{\to} \mathrm{Spc}$$

is not convergent.

Remark 6.1.12 ([35]*Remark 17.3.2.2)**.** The inclusion

$$^{\mathrm{conv}}\mathrm{PreStk} \hookrightarrow \mathrm{PreStk}$$

of the subcategory of convergent prestacks into all prestacks has a left adjoint $(-)^{\mathrm{conv}}$ given informally by

$$(\mathscr{X})^{\mathrm{conv}}(S) \simeq \lim_{n\geq 0} \mathscr{X}(\tau^{\leq n}(S)).$$

The conditions of §4.2.2 easily generalize to prestacks.

Definition 6.1.13.

- Let $\mathscr{Y} \in {}^{\leq n}\mathrm{PreStk}$, then $\mathscr{Y}$ is said to be *locally of finite type* if the canonical map

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}(\mathscr{Y}|_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}) \to \mathscr{Y}$$

is an isomorphism.

- For $\mathscr{X} \in \mathrm{PreStk}$ one says that $\mathscr{X}$ is locally almost of finite type if
  (i) $\mathscr{X}$ is convergent;
  (ii) for all $n \geq 0$, ${}^{\leq n}\mathscr{X}$ is locally of finite type.

We will denote the subcategory of prestacks locally almost of finite type by $\mathrm{PreStk}_{\mathrm{laft}}$.

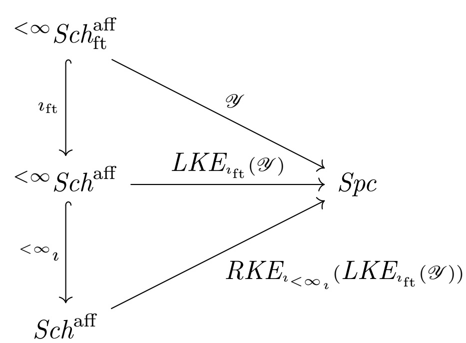
The following result is a type of sanity check.

Proposition 6.1.14. *(i) $\mathscr{Y} \in {}^{\leq n}PreStk$ is locally of finite type if and only if $\mathscr{Y} : ({}^{\leq n}Sch^{\mathrm{aff}})^{\mathrm{op}} \to Spc$ sends co-filtered limits to colimits;*

(ii) $S \in {}^{\leq n}Sch^{\mathrm{aff}}$ is of finite type if and only if $h_S := Maps_{{}^{\leq n}Sch^{\mathrm{aff}}}(-, S) \in {}^{\leq n}PreStk$ is locally of finite type;

(iii) $Sch^{\mathrm{aff}}_{\mathrm{ft}} \simeq Sch^{\mathrm{aff}} \cap PreStk_{\mathrm{laft}}$, where Sch^{aff} is identified with a subcategory of PreStk via the Yoneda embedding.

Proposition 6.1.15. *Consider the diagram*



A prestack $\mathscr{X}$ is locally almost of finite type if and only if the following two canonical maps[3]

$$\mathscr{X} \to RKE_{^{<\infty}\imath}(\mathscr{X}|_{^{<\infty}Sch^{\mathrm{aff}}}) \leftarrow RKE_{^{<\infty}\imath}(LKE_{\imath_{\mathrm{ft}}}(\mathscr{X}|_{^{<\infty}Sch^{\mathrm{aff}}_{\mathrm{ft}}}))$$

are isomorphisms.

[3]Notice that since $\mathrm{RKE}_{^{<\infty}\imath}$ is fully faithful the condition that the left arrow is an isomorphism is equivalent to

$$\mathrm{LKE}_{\imath_{\mathrm{ft}}}(\mathscr{X}|_{^{<\infty}\mathrm{Sch}^{\mathrm{aff}}}) \to \mathscr{X}|_{^{<\infty}\mathrm{Sch}^{\mathrm{aff}}}$$

being an isomorphism.

Exercise 6.1.16. Prove Propositions 6.1.14 and 6.1.15. (Hint: See [16, Chapter 2, §1].)

For any $k \geq 0$ let $P_{\leq k} : \mathrm{Spc} \overset{\tau_{\leq k}}{\to} \mathrm{Spc}^{\leq k} \to \mathrm{Spc}$ denote the composite of the truncation to k-truncated spaces composed with the canonical inclusion.

Definition 6.1.17. A prestack $\mathscr{X} \in {}^{\leq n}\mathrm{PreStk}$ is said to be *k-truncated* if it factors as follows:

$$\begin{array}{ccc} ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} & \dashrightarrow & \mathrm{Spc}^{\leq k} \\ & & \downarrow \\ & & \mathrm{Spc} \end{array}$$

We will denote the category of k-truncated prestacks in n-coconnective affine schemes by ${}^{\leq n}\mathrm{PreStk}_{\leq k}$.

Example 6.1.18.
(i) For any $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, the prestack $h_S \in {}^{\leq n}\mathrm{PreStk}$ is n-truncated.

(ii) For any topological space $X \in \mathrm{Spc}$, let

$$\underline{X}(S) := X$$

denote the associated constant prestack in ${}^{\leq n}\mathrm{PreStk}$. Then $\underline{X}$ is k-truncated if and only if X is k-truncated.

Remark 6.1.19. The category ${}^{c\ell}\mathrm{PreStk}_{\leq 1}$ is the 1-category of classical prestacks and ${}^{c\ell}\mathrm{PreStk}_{\leq 0}$ is the 1-category of presheaves of sets on ordinary affine schemes, i.e. functors of point in Grothendieck's approach to algebraic geometry.

Remark 6.1.20. Something about k-truncated prestacks not being k-truncated objects in PreStk and what one means by k-truncated when n is not given.

6.2 Topologies on affine schemes

Definition 6.2.1. A morphism $f : T = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$ in $\mathrm{Sch}^{\mathrm{aff}}$ is *flat* if the corresponding morphism in CAlg is flat, i.e. B is a flat A-module, that is[4]

(i) $H^0(B)$ is a flat $H^0(A)$-module;

(ii) for every $i \in \mathbb{Z}$ one has
$$H^i(B) \simeq H^i(A) \otimes_{H^0(A)} H^0(B);$$

(ii)' for every $M \in \mathrm{Mod}_A$
$$H^i(B \otimes_A M) \simeq H^0(B) \otimes_{H^0(A)} H^i(M);$$

(ii)" for every $N \in \mathrm{Mod}_A^{\heartsuit}$
$$B \otimes_A N \in \mathrm{Mod}_B^{\heartsuit}.$$

Notice that given $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ and an affine scheme S' with a flat morphism $S' \to S$, then $S' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

Exercise 6.2.2.
(i) Consider $f : T \to S$ a flat morphism, then $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ implies $T \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

(ii) A morphism $f : T \to S$ is flat if and only if for all $n \geq 0$ one has ${}^{\leq n}f : {}^{\leq n}T \to {}^{\leq n}S$ is flat.

We now collect a number of useful notions for a morphism between affine schemes:

Definition 6.2.3. A morphism $f : T \to S$ between affine scheme is

- *flat of finite presentation* (ppf)[5] if f is flat and ${}^{c\ell}f : {}^{c\ell}T \to {}^{c\ell}S$ is of finite presentation;

[4]Here one can take either (ii), (ii)' or (ii)" together with (i).
[5]The abbreviation stands for "plat de presentation finie".

- *smooth* if f is flat and ${}^{c\ell}f$ is smooth;
- *étale* if f is flat and ${}^{c\ell}f$ is étale;
- *open embedding* if f is flat and ${}^{c\ell}f$ is open embedding[6];
- *Zariski* if f is flat and ${}^{c\ell}f$ is a disjoint union of open embeddings.

Question: Is it the case that ppf and epimorphism in the ∞-category CAlg implies that $f : A \to B$ is open?

The next result makes precise, at least at the level of affine schemes (a more general result is also true, see **??** below), that the extra derived structure doesn't change the underlying topology of an affine scheme, or more generally its underlying étale topos.

Proposition 6.2.4. *Let S be an affine schemes, the pullback functor*

$$Sch^{\mathrm{aff}}_{/S} \to Sch^{\mathrm{aff}}_{/^{c\ell}S}$$
$$(T \to S) \mapsto T \underset{S}{\times} {}^{c\ell}S$$

gives equivalences between the subcategories

$$\{f : S' \to S \mid f \text{ étale }\} =: Sch^{\mathrm{aff}}_{\text{étale over } S} \simeq Sch^{\mathrm{aff}}_{\text{étale over } {}^{c\ell}S} := \{f : S'_0 \to {}^{c\ell}S \mid f \text{ étale }\}$$

and

$$\{f : S' \to S \mid f \text{ open embedding }\} =: Sch^{\mathrm{aff}}_{\text{open in } S} \simeq Sch^{\mathrm{aff}}_{\text{open in } {}^{c\ell}S} := \{f : S'_0 \to {}^{c\ell}S \mid f \text{ open embedding }\}.$$

Idea of proof. Step 1: For each $n \geq 0$ establish an equivalence

$$\mathrm{Sch}^{\mathrm{aff}}_{\text{étale over } \tau^{\leq(n+1)}(S)}\mathrm{Sch}^{\mathrm{aff}}_{\text{étale over } \tau^{\leq n}(S)}$$
$$S'_{n+1} \to \tau^{\leq(n+1)}(S) \mapsto S'_{n+1} \underset{\tau^{\leq(n+1)}(S)}{\times} \tau^{\leq n}(S).$$

using the deformation theory. By (**??**) one has an equivalence

$$\{S'_{n+1} \in {}^{\leq(n+1)}\mathrm{Sch}^{\mathrm{aff}} \mid {}^{\leq n}S'_{n+1} \simeq {}^{\leq n}S\}$$
$$\simeq \{\text{square-zero extensions of } {}^{\leq n}S \text{ by } \mathscr{I} \in \mathrm{QCoh}({}^{\leq n}S)[n+1]\}m,$$

where by definition the right-hand side is given by a morphism in $\mathrm{Hom}_{\mathrm{QCoh}(^{\leq n}S)}(T^*(S'_{n+1}/\leq S), \mathscr{I}[1])$. Then **??** implies that $T^*(S'_{n+1}/^{\leq n}S)$ vanishes, since $S'_{n+1} \to {}^{\leq n}S$ is étale. *Step 2:* There is an equivalence

$$\mathrm{Sch}^{\mathrm{aff}}_{\text{étale over } S} \to \lim_{n\geq 0}\mathrm{Sch}^{\mathrm{aff}}_{\text{étale over } \tau^{\leq n}(S)}.$$

This is immediate from the fact that any morphism between affine schemes is convergent. □

To define a topology we need to specify when a morphism is a covering the above Proposition suggests that it is enough to impose a requirement on the classical part.

Definition 6.2.5. A morphism $f : S' \to S$ is a *flat* (resp. *ppf, smooth, étale,* or *Zariski*) *covering* if f is a flat (resp. ppf, smooth, étale, or Zariski) morphism and ${}^{c\ell}f$ is surjective.

These properties have the usual behavior with respect to base change and pullback and are local when expected to be so. For instance, let $S' \to S$ be a covering for a certain topology σ and $f : T \to S$ an arbitrary morphism, if $S' \underset{S}{\times} T \to T$ is τ, where τ is σ or coarser, then f is also τ. Check this is true!

Another example, given $f : T \to S$ and a τ-covering $T' \to T$, then if the composite $T' \to S$ is τ, then so is f.

The next definition singles out the first class of morphisms between prestacks to which we can extend the topological properties of morphisms just discussed.

[6] Recall that means the associate morphism of algebras $A \to B$ is ppf and an epimorphism in the category of discrete rings.

Definition 6.2.6.
- A morphism $f : \mathscr{X} \to \mathscr{Y}$ of prestacks is said to be *affine schematic* if for every affine scheme $S \to \mathscr{Y}$ the fiber product $S \underset{\mathscr{Y}}{\times} \mathscr{X}$ is an affine scheme.
- Given an affine schematic morphism $f : \mathscr{X} \to \mathscr{Y}$ we say that f is *flat* (resp. *ppf, smooth, étale, open embedding* or *Zariski*) if for every affine point $S \to \mathscr{Y}$ the morphism
$$S \underset{\mathscr{Y}}{\times} \mathscr{X} \to S$$
is flat (resp. ppf, smooth, étale, open embedding, or Zariski).

6.3 Stacks

We first introduce a piece of convenient notation: given a morphism $f : S' \to S$ in the category of affine schemes the *Čech nerve of* f is the simplicial object
$$(S'/S)^{\bullet} : \Delta^{\mathrm{op}} \to \mathrm{Sch}^{\mathrm{aff}}$$
whose first level are given as follows:
$$\cdots \rightrightarrows S' \underset{S}{\times} S' \underset{S}{\times} S' \rightrightarrows S' \underset{S}{\times} S' \rightrightarrows S'$$
where the morphisms are the canonical projections.

Definition 6.3.1. A prestack $\mathscr{X}$ *satisfy flat* (resp. *ppf, étale, smooth* or *Zariski*) *descent* if for every flat (resp. ppf, étale, smooth or Zariski) covering $S' \to S$ the canonical map
$$\mathscr{X}(S) \to \lim_{\Delta^{\mathrm{op}}}\big(\mathscr{X}(S') \rightrightarrows \mathscr{X}(S') \underset{\mathscr{X}(S)}{\times} \mathscr{X}(S') \rightrightarrows \cdots \big)$$
is an isomorphism. We will say a prestack is a *stack* if it satisfies descent for the étale topology and we let Stk denote the subcategory of stacks.

Remark 6.3.2. One could have considered different topologies, but in a certain sense the choice of the étale topology gives essentially the most general notion, once one imposes the condition of being an Artin stack, see below Reference ? for more details on this.

Lemma 6.3.3. *The canonical inclusion Stk $\hookrightarrow$ PreStk has a left adjoint L : PreStk $\to$ Stk called the* sheafification functor. *Moreover, L is left exact, i.e. commutes with finite limits.*

Proof. The main idea is that L is obtained by inverting *étale equivalences*, i.e. morphisms $f : \mathscr{X}_1 \to \mathscr{X}_2$ in PreStk such that
$$\mathrm{Hom}_{\mathrm{PreStk}}(\mathscr{X}_2, \mathscr{Y}) \to \mathrm{Hom}_{\mathrm{PreStk}}(\mathscr{X}_1, \mathscr{Y})$$
is an equivalence for all $\mathscr{Y} \in \mathrm{Stk}$. □

Notation 6.3.4. In [16] the authors usually consider any object as a prestack, in particular for stacks they denote by L the composite $\mathrm{PreStk} \xrightarrow{L} \mathrm{Stk} \hookrightarrow \mathrm{PreStk}$. We make follow the same notion below. Is this true? Decide on it!

Proposition 6.3.5. *Let $R \to R'$ be a flat morphism of derived rings, one has an equivalence of categories*
$$Mod_R \to \lim_{\Delta^{\mathrm{op}}}\big(Mod_{R'} \rightrightarrows Mod_{R' \otimes_R R'} \rightrightarrows \cdots \big).$$

Proof. *Argument 1:* One reduces the statement to the connective objects, i.e. it is enough to prove that
$$\mathrm{Mod}_R^{\leq 0} \to \lim_{\Delta^{\mathrm{op}}}\big(\mathrm{Mod}_{R'}^{\leq 0} \rightrightarrows \mathrm{Mod}_{R' \otimes_R R'}^{\leq 0} \rightrightarrows \cdots \big),$$
is an equivalence. Use Bar–Beck–Lurie to obtain conditions that can be checked by hand on the monad $\mathrm{oblv}_{R' \to R} \circ (-) \otimes_R R' : \mathrm{Mod}_R \to \mathrm{Mod}_R$, where $\mathrm{oblv}_{R' \to R} : \mathrm{Mod}_{R'} \to \mathrm{Mod}_R$ forgets the R'-module structure to an R-module structure. The necessary condition is equivalent to (...).

Argument 2: One needs to argue the following three points:

- the category $\lim_{\Delta}(\mathrm{Mod}_{(R'/R)^{\bullet}})$ has a t-structure;
- the functors $\mathrm{Mod}_R \rightleftarrows \lim_{\Delta}(\mathrm{Mod}_{(R'/R)^{\bullet}})$ are t-exact;
- reduce to a statement about the heart of the t-structure, which is equivalent to the usual flat descent for discrete modules over discrete rings.

□

Example 6.3.6. The prestack

$$\mathscr{Q} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \overset{\mathrm{QCoh}_{(-)^*}}{\to} \mathrm{Cat}_\infty \overset{(-)^{\simeq}}{\to} \mathrm{Spc}$$

satisfies flat descent. In particular, $\mathscr{Q}$ is a stack.

We would expect that for reasonable topologies an affine scheme, when seen as a prestack, satisfy descent. The following check that the étale topology is sub-canonical.

Proposition 6.3.7. *Given an affine scheme T the corresponding prestack h_T satisfies étale descent.*

Proof. Let $S' \to S$ be an étale cover we have a diagram

$$\mathrm{Hom}_{\mathrm{Sch}^{\mathrm{aff}}}(S,T) \longrightarrow \mathrm{Hom}_{\mathrm{Sch}^{\mathrm{aff}}}(S',T) \rightrightarrows \mathrm{Hom}_{\mathrm{Sch}^{\mathrm{aff}}}(S' \underset{S}{\times} S',T) \Rrightarrow \cdots . \tag{6.1}$$

Let $T = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(A')$ for derived rings B, A and A', then equation (6.1) corresponds to

$$\mathrm{Hom}_{\mathrm{CAlg}}(B,A) \longrightarrow \mathrm{Hom}_{\mathrm{CAlg}}(B,A') \rightrightarrows \mathrm{Hom}_{\mathrm{CAlg}}(B,A' \otimes_A A') \Rrightarrow \cdots .$$

Formally one has an equivalence

$$\mathrm{Tot}(\mathrm{Hom}_{\mathrm{CAlg}}(B,(A'/A)^{\bullet})) \to \mathrm{Hom}_{\mathrm{CAlg}}(B,\mathrm{Tot}((A'/A)^{\bullet})).$$

Since the functor $\mathrm{oblv} : \mathrm{CAlg} \to \mathrm{Vect}$ that forgets the structure of a commutative algebra object preserves limits (by []). Proposition 6.3.5 implies that $\mathrm{Tot}((A'/A)^{\bullet}) \simeq A$, which gives that

$$\mathrm{Tot}(\mathrm{Hom}_{\mathrm{CAlg}}(B,(A'/A)^{\bullet})) \simeq \mathrm{Hom}_{\mathrm{CAlg}}(B,A)$$

as we needed to prove. □

The localization functor $L : \mathrm{PreStk} \to \mathrm{Stk} \to \mathrm{PreStk}$ is hard to control or describe very explicitly. The following notion helps one understands a bit more what is happening.

Definition 6.3.8. A morphism of prestacks $f : \mathscr{X} \to \mathscr{Y}$ is an *étale surjection* if for all affine points $y : S \to \mathscr{Y}$ there exists an étale cover $\varphi : S' \to S$ such that

$$\varphi^*(y) : S' \to \mathscr{Y} \text{ belongs to the essential image of } f(S') : \mathscr{X}(S') \to \mathscr{Y}(S').$$

Here is an example of how the notion of étale surjection is used.

Proposition 6.3.9. *Let $\mathscr{X} \to \mathscr{Y}$ be an étale surjection then*

$$|\mathscr{X}^{\bullet}/\mathscr{Y}|_{PreStk} \to \mathscr{Y}$$

is an étale equivalence, i.e.

$$|L(\mathscr{X})^{\bullet}/L(\mathscr{Y})|_{Stk} \simeq |L(\mathscr{X}^{\bullet}/\mathscr{Y})|_{Stk} L(|\mathscr{X}^{\bullet}/\mathscr{Y}|_{PreStk}) \to L(\mathscr{Y}) \tag{6.2}$$

is an isomorphism.

Proof. Write an argument for this. □

Include in this section some parts of the discussion in the note derived quot scheme. In particular, prop:epimorphism-local-surjection is relevant for this.

Corollary 6.3.10. *For $f : \mathscr{X} \to \mathscr{Y}$ an affine schematic morphism between prestacks if f is flat (resp. ppf, smooth, étale, Zariski or open embedding) then so is $L(f) : L(\mathscr{X}) \to L(\mathscr{Y})$.*

Conditions on stacks

In this section we address the following **Question:** How does descent relate to the conditions of n-coconnective, convergent, locally (almost) of finite type and truncatedness introduced in Section 6.1?

Given an object $\mathscr{X} \in {}^{\leq n}\mathrm{PreStk}$ we will say that $\mathscr{X}$ satisfies étale descent, if for every étale cover $S' \to S$ in ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ one has an isomorphism

$$\mathscr{X}(S) \to \mathrm{Tot}(\mathscr{X}(S'^{\bullet}/S)).$$

We will let

$${}^{\leq n}\mathrm{Stk} := \{\mathscr{X} \in {}^{\leq n}\mathrm{PreStk} \mid \mathscr{X} \text{ satisfies étale descent }\},$$

denote the subcategory of ${}^{\leq n}\mathrm{PreStk}$ consisting of étale sheaves and ${}^{\leq n}L : {}^{\leq n}\mathrm{PreStk} \to {}^{\leq n}\mathrm{Stk} \to {}^{\leq n}\mathrm{PreStk}$ the corresponding localization functor.

When an $\mathscr{X} \in {}^{\leq n}\mathrm{PreStk}$ is k-truncated for some $k \geq 0$ one can describe the sheafification functor more explicitly. Consider the endofunctor $(-)^{\dagger} : {}^{\leq n}\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$ given by

$$\mathscr{X}^{\dagger}(S) := \underset{S' \to S \text{étale cover}}{\mathrm{colim}} \mathscr{X}(\mathrm{Tot}(\mathscr{X}(S'^{\bullet}/S))).$$

Lemma 6.3.11. *Assume that $\mathscr{X} \in {}^{\leq n}PreStk$ is k-truncated for some $k \geq 0$, then*

$${}^{\leq n}\mathscr{X} \simeq \mathscr{X}^{\dagger^{(k+2)}}.$$

In particular, ${}^{\leq n}\mathscr{X}$ is k-truncated Justify this!.

Remark 6.3.12. If $\mathscr{X}$ is convergent it might be tricky to check if $L(\mathscr{X})$ is convergent. However, if $\mathscr{X} \in \mathrm{Stk}$ then ${}^{\mathrm{conv}}\mathscr{X}$ (see Remark 6.1.12) is a stack.

Warning 6.3.13. The left Kan extension

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}$$

does not send ${}^{\leq n}\mathrm{Stk}$ to Stk, since we are trying to commute a Totalization with an arbitrary colimit. Even in the simplest examples it is unclear that the result is a stack, for instance for $\mathbb{P}^1$ considered as a classical prestack, i.e. $\mathbb{P}^1 : {}^{c\ell}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}$ it is not known if

$$\mathrm{LKE}_{{}^{c\ell}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathbb{P}^1)$$

satisfies étale descent.

The following notion formalizes the notion of an n-coconnective stack.

Definition 6.3.14. A stack $\mathscr{X}$ is *n-coconnective* if the map

$$L \circ \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}({}^{\leq n}\mathscr{X}) \to \mathscr{X}$$

is an equivalence.

In particular, a 0-coconnective stack (which some places refer to as a "classical" stack) is *not* recovered from classical affine schemes simply by left Kan extension to affine schemes via ${}^{c\ell}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$, one also needs to sheafify the resulting prestack in *derived* affine schemes.

Remark 6.3.15. It is clear that an n-coconnective prestack that is also a stack is an n-coconnective stack.

Remark 6.3.16. Simlar to the n-coconnective condition, the locally of finite type condition also interacts subtly with the descent condition. We refer the reader to Chapter 2, §2.7 in [16] for a detailed discussion.

Example 6.3.17. Consider T an affine scheme, 6.3.7 implies that h_T is a stack. We would like to understand when is h_T a 0-coconnective stack. We will tackle the easier question of when is h_T 0-coconnective as a prestack, which implies the later. Is it equivalent in this case? Let $T = \mathrm{Spec}(B)$, we consider

$$\begin{aligned}\mathrm{LKE}_{^{c\ell}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} {}^{c\ell}T(S) &\simeq \operatorname*{colim}_{S \to S_0 \mid S_0 \in {}^{c\ell}\mathrm{Sch}^{\mathrm{aff}}} {}^{c\ell}T(S_0) \\ &\simeq \operatorname*{colim}_{A_0 \to A \mid A_0 \in \mathrm{CAlg}^{\mathrm{disc.}}} \mathrm{Hom}_{\mathrm{CAlg}^{\mathrm{disc.}}}(\tau^{\geq 0}(B), A_0) \\ &\simeq \operatorname*{colim}_{A_0 \to A \mid A_0 \in \mathrm{CAlg}^{\mathrm{disc.}}} \mathrm{Hom}_{\mathrm{CAlg}}(B, A_0),\end{aligned}$$

where we denote $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A_0)$. Since any $A \in \mathrm{CAlg}$ is a sifted colimit of discrete algebras, from its presentation as a simplicial commutative ring, so if B is a compact and projective object of CAlg then we have an isomorphism

$$\operatorname*{colim}_{A_0 \to A \mid A_0 \in \mathrm{CAlg}^{\mathrm{disc.}}} \mathrm{Hom}_{\mathrm{CAlg}}(B, A_0) \simeq \mathrm{Hom}_{\mathrm{CAlg}}(B, A).$$

By Proposition 4.2.7 one has that $B \simeq \mathrm{Sym}(V)$ for $V \in \mathrm{Vect}^{\leq 0}$ a perfect object, i.e. V is a finite direct sum of $k[m]$ for some $m \geq 0$.

6.4 Schemes

Before discussing deformation theory we will introduce a class of prestacks that naturally generalizes classical schemes. The philosophy here is that (derived) schemes are prestacks with certain properties, so we don't need to give any extra data–this is in contrast with the approach taken in [35] where one defines (derived) schemes as a certain kind of locally ringed ∞-topos.

Figure out how much more about locally ringed ∞-topoi do I want to say.

Definition 6.4.1. A prestack Z is said to be a *scheme* if

(i) Z satisfies étale descent;

(ii) the diagonal morphism $Z \to Z \times Z$ is affine schematic and for every $T \to Z \times Z$ the map

$$^{c\ell}\left(T \underset{Z \times Z}{\times} Z\right) \to {}^{c\ell}T$$

is a closed embedding;

(iii) (*Zariski atlas*) there exists a collection of morphism $\{f_i : S_i \to Z\}_I$ such that

- each f_i is an open embedding;
- for every affine scheme $T \to Z$ the following morphism of spaces

$$\bigsqcup_I {}^{c\ell}\left(T \underset{Z}{\times} S_i\right) \to {}^{c\ell}T.$$

induces a surjection on connected components.

We let $\mathrm{Sch} \hookrightarrow \mathrm{Stk} \hookrightarrow \mathrm{PreStk}$ denote the subcategory of schemes.

Remark 6.4.2. We could have either imposed a weaker version of (i) that Z satisfy Zariski descent or a stronger version of (i) that Z satisfy flat descent and we still would obtain the same class of objects. Give an argument for this.

Remark 6.4.3. For ease of exposition we are restricting in Definition 6.4.1 to the derived generalization of separated classical schemes by imposing condition (ii). If we omit condition (ii) we don't have Lemma 6.4.9 below and the comparison with the usual notion of classical schemes is a bit more convoluted. For completeness the case of non-separated (derived) schemes will be included in 1-geometric stacks defined below reference to section later.

Let's discuss some formal properties of the definition.

Lemma 6.4.4. *For Z a scheme and $\{S_i \to Z\}_I$ a Zariski atlas, the morphism $f : \sqcup_I S_i \to Z$ is an étale surjection. Thus, Z is obtained from the geometric realization*[7] *of the Čech nerve of f, i.e. the map*

$$L\left(\left|(\bigsqcup_I S_i)^\bullet/Z\right|_{\mathrm{PStk}}\right) \to Z \tag{6.3}$$

is an isomorphism.

Proof. Let $T \to \sqcup_I S_i$ be an affine scheme, notice that for the morphism

$$(\bigsqcup_I S_i)(T) \to Z(T)$$

to be fully faithful is equivalent to

$$\pi_0((\bigsqcup_I S_i)(T)) \to \pi_0(Z(T))$$

being surjective, since for any prestack $\mathscr{X}$ one has

$$\pi_0(\mathscr{X}(T)) \simeq \pi_0(\mathscr{X}({}^{\mathrm{c}\ell}T)),$$

the claim follows directly from the second condition of (iii).

The second statement is a consequence of Proposition 6.3.9. □

The presentation of (6.3) is useful to determine when one can find a derived enhancement to a classical scheme.

Proposition 6.4.5. *Suppose we are given the data of a groupoid-object $S^\bullet$ in PreStk and let*

$$Z := L(|S^\bullet|)$$

be the sheafification of its geometric realization. Then to check that Z is a derived scheme it is enough to check the following:

(a.) $S^0 = \sqcup_I S^0_i$ and $S^1 = \sqcup_J S^1_j$ disjoint unions of affine schemes, a morphism $S^1 \to S^0$ such that the restriction $S^1_j \to S^0_i$ is an open embedding for each $i \in I$ and $j \in J$;

(b.) ${}^{\mathrm{c}\ell}Z$ a classical scheme and the induced morphism $\sqcup_I {}^{\mathrm{c}\ell}S^0_i \to {}^{\mathrm{c}\ell}Z$ is a (classical) Zariski atlas.

Then Z is a scheme and the maps $\sqcup_I S^0_i \to Z$ give a Zariski atlas of Z.

Proof. We claim that it is enough to check that each $S^0_i \to Z$ is an open embedding, since the surjectivity condition of Definition 6.4.1 (iii) follows immediately from the hypothesis that the underlying classical morphism is a classical Zariski atlas. Next we notice that for a morphisms of stacks $L(\mathscr{X}) \to L(\mathscr{Y})$ to be schematic it is enough that the morphism $\mathscr{X} \to \mathscr{Y}$ of prestacks is schematic, thus it is enough to check that $S^0_i \to |S^\bullet|$ is an affine open embedding, where the geometric realization is taken in the category of prestacks. Notice that given $T \in \mathrm{Sch}^{\mathrm{aff}}$ and a morphism $T \to |S^\bullet|$ this factors as follows:

$$T \to S^0 \to |S^\bullet|,$$

[7] Taken in the category of étale stacks.

since colimits of prestacks are computed point-wise, i.e. $|S^\bullet|(T) \simeq \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} S^n(T)$. So one has

$$T \underset{|S^\bullet|}{\times} S_i^0 \simeq T \underset{S^0}{\times} S^0 \underset{|S^\bullet|}{\times} S_i^0.$$

Thus, it is enough to prove that $S^0 \underset{|S^\bullet|}{\times} S_i^0 \to S_i^0$ is an affine open embedding. However this follows from

$$S^0 \underset{|S^\bullet|}{\times} S_i^0 \simeq \left(S^0 \underset{|S^\bullet|}{\times} S^0\right) \underset{S^0}{\times} S_i^0 \simeq S^1 \underset{S^0}{\times} S_i^0$$

and the assumption (a.). ☐

In practice, what can be tricky is the construction of the groupoid-object $S^\bullet$, in fact the main tool that one has is to specify a morphism $S^0 \to Z$ and to define $S^\bullet$ as its Čech nerve (see Example 2.4.4).

Example 6.4.6. What is the problem with this construction? Suppose we are given $S^1 \to S^0$ a morphism between disjoint union of affine schemes. Consider the following pushout diagram

$$\begin{array}{ccc} {}^{\mathrm{c}\ell}S^1 & \longrightarrow & {}^{\mathrm{c}\ell}S^0 \\ \downarrow & & \downarrow \\ {}^{\mathrm{c}\ell}S^0 & \longrightarrow & Z_0 \end{array}$$

which defines a classical scheme Z_0. Assume that S^1 and S^0 are 0-coconnective, i.e. the canonical maps

$$\mathrm{LKE}_{{}^{\mathrm{c}\ell}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}({}^{\mathrm{c}\ell}S^0) \to S^0 \quad \text{and} \quad \mathrm{LKE}_{{}^{\mathrm{c}\ell}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}({}^{\mathrm{c}\ell}S^1) \to S^1$$

are equivalences. To be continued.

The Proposition above together with the equivalence of the étale topos over an affine scheme with the étale topos over its underlying classical affine scheme gives the following result.

Given a scheme $Z \in \mathrm{Sch}$ let $\mathrm{Sch}_{\mathrm{aff.Zar.inZ}}$ denote the subcategory of $\mathrm{Sch}_{/Z}$ spanned by morphisms $f : Z' \to Z$ which are affine and Zariski[8]. Similarly, given a classical scheme $Z_0 \in {}^{\mathrm{c}\ell}\mathrm{Sch}$ let ${}^{\mathrm{c}\ell}\mathrm{Sch}_{\mathrm{aff.Zar.inZ_0}}$ denote the the subcategory of ${}^{\mathrm{c}\ell}\mathrm{Sch}_{/Z_0}$ spanned by morphisms $f : Z_0' \to Z_0$ which are disjoint union of open embeddings.

Corollary 6.4.7. *For any scheme $Z \in Sch$ the functor of passing to the underlying classical scheme gives an equivalence*

$$^{\mathrm{c}\ell}(-) : Sch_{\mathrm{aff.Zar.inZ}} \to {}^{\mathrm{c}\ell}Sch_{\mathrm{aff.Zar.in}^{\mathrm{c}\ell}\mathrm{Z}}.$$

Moreover, this functor is part of an adjunction $({}^{\mathrm{c}\ell}-, ?)$, and it takes open embeddings to open embeddings.

The following are straight-forward consequences of Corollary 6.4.7.

Corollary 6.4.8. *Given a scheme $Z \in Sch$ we have*

(i) given $f : Z' \to Z$ with f affine Zariski, then Z' is an affine scheme if and only if ${}^{\mathrm{c}\ell}Z'$ is a classical affine scheme;

(ii) Z is an affine scheme if and only if ${}^{\mathrm{c}\ell}Z$ is an affine scheme.

The following is a sanity check that our definition of schemes when restricted to classical objects obtains the usual definition.

Lemma 6.4.9. *Let $Z_0 \in {}^{\mathrm{c}\ell}PreStk$ be a classical prestack, such that*

(i) Z_0 satisfies étale descent;

[8] Recall that one says a map between prestacks is Zariski, if it is affine representable and the underlying morphism between classical affine schemes is a disjoint union of open embeddings.

(ii) the diagonal morphism $Z_0 \to Z_0 \times Z_0$ is a closed embedding, i.e. for any classical affine scheme $T_0 \to Z_0 \times Z_0$ the base change morphism $Z_0 \underset{Z_0 \times Z_0}{\times} T_0 \to T_0$ is a closed embedding;

(iii) there exists a Zariski cover $\sqcup_I S_i \to Z_0$, where each S_i is a classical affine scheme and $Z_i \to Z_0$ is an open embedding.

Then Z_0 is a classical (separated) scheme. In particular, Z_0 is 0-truncated.

Proof. Let $S^\bullet$ denote the simplicial object obtained by taking the Čech nerve of $S^0 := \sqcup_I S_i \to Z_0$, we need to check that

$$Z_0 \simeq \mathrm{L}(|S^\bullet|_{\mathrm{PreStk}})$$

is 0-truncated. By write this result it is enough to check that $|S^\bullet|_{\mathrm{PreStk}}$ is 0-truncated. Given a classical affine scheme T_0, by definition we have

$$|S^\bullet|_{\mathrm{PreStk}}\,(T_0) \simeq |S^\bullet(T_0)|_{\mathrm{Spc}}\,.$$

We now notice that $|S^\bullet(T_0)|_{\mathrm{Spc}}$ satisfies the Kan condition, i.e. for every $n \geq 1$ and $0 < i < n$ the morphism

$$|S^\bullet(T_0)|_{\mathrm{Spc}}\,(\Delta^n) \to |S^\bullet(T_0)|_{\mathrm{Spc}}\,(\Lambda^n_i)$$

induces a surjection on π_0. In fact, it is actually a homotopy equivalence, since $|S^\bullet(T_0)|_{\mathrm{Spc}}$ is a groupoid object in Spc. Thus, one has

$$\pi_n\left(|S^\bullet(T_0)|_{\mathrm{Spc}}\right) = \pi_0\left(\mathrm{Fib}(|S^\bullet(T_0)|_{\mathrm{Spc}}\,(\Delta^n) \to |S^\bullet(T_0)|_{\mathrm{Spc}}\,(\partial\Delta^n))\right).$$

In particular, one obtains that

$$\pi_1\left(|S^\bullet(T_0)|_{\mathrm{Spc}}\right) = \pi_0\left(\mathrm{Fib}(\mathrm{Hom}_{^{c\ell}\mathrm{Sch}^{\mathrm{aff}}}(T_0, S^1) \xrightarrow{\alpha} \mathrm{Hom}_{^{c\ell}\mathrm{Sch}^{\mathrm{aff}}}(T_0, S_0 \times S_0)\right),$$

where $S^1 := S^0 \underset{Z^0}{\times} S^0$. However, α is injective, since $Z_0 \to Z_0 \times Z_0$ closed implies that $S^1 \to S^0 \times S^0$ is locally closed. Similarly, we can conclude that $\pi_n\left(|S^\bullet(T_0)|_{\mathrm{Spc}}\right)$ vanishes for $n \geq 2$. □

Remark 6.4.10. For the reader that has in mind a definition of classical schemes using locally ringed spaces here is a sketch of how the definition in Lemma 6.4.9. See [13, Chapter 1, §4.4, Comparison Theorem] where one has that a functor $\mathscr{X}_0 : {}^{c\ell}\mathrm{Sch}^{\mathrm{aff}} \to \mathbf{Set} \simeq \mathrm{Spc}^{\leq 0}$ is equivalent to a scheme defined as a locally ringed topological space, e.g. [], if and only if $\mathscr{X}_0$ satisfies:

(i) any classical affine scheme $S_0 = \mathrm{Spec}(R_0)$ and $\{f_1, \dots, f_n\}$ a finite set of elements in R_0 such that $(f_1, \dots, f_n) = R_0$ the following canonical map

$$\mathscr{X}_0(S_0) \to \lim\left(\prod_{i=1}^n \mathscr{X}_0(\mathrm{Spec}(R_{0,f_i})) \rightrightarrows \prod_{i,j=1}^n \mathscr{X}_0(\mathrm{Spec}(R_{0,f_if_j}))\right)$$

is an equivalence;

(ii) there exists a collection $(U_i)_I$ of affine open subfunctors of $\mathscr{X}_0$ such that for every field k the following holds

$$\bigcup_I U_i(\mathrm{Spec}(k)) \xrightarrow{\simeq} \mathscr{X}_0(\mathrm{Spec}(k)).$$

Exercise 6.4.11. Check that conditions (i) and (ii) of Remark 6.4.10 imply conditions (i) and (iii) of Definition 6.4.1[9]

[9] As explained in Remark 6.4.3 condition (ii) is just imposed so that we automatically obtain that the classical truncation of a derived scheme is 0-truncated and hence a separated classical scheme.

Definition 6.4.12. Given a scheme Z we will say that Z is n-coconnective if either of the following equivalent conditions hold:

(i) the canonical morphism
$$\mathrm{L} \circ \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}({}^{\leq n}Z) \to Z$$
is an isomorphism;

(ii) there exists a Zariski atlas $\sqcup_I S_i \to Z$ where each S_i is n-coconnective.

We will denote by ${}^{\leq n}\mathrm{Sch}$ the category of n-coconnective schemes.

In fact the following more general statement is true and can be proved as Lemma 6.4.9.

Lemma 6.4.13. *Given an n-coconnective scheme $Z \in {}^{\leq n}Sch$ and a point $z : T_0 \to Z$ from a classical affine scheme T_0. For any morphism of affine schemes $T_0 \to T$ where $T \in {}^{\leq n}Sch^{\mathrm{aff}}$ and ${}^{\mathrm{c}\ell}T_0 \overset{\simeq}{\to} {}^{\mathrm{c}\ell}T$ the space of lifts*

$$\begin{array}{ccc} T & & \\ \uparrow & \searrow & \\ T_0 & \longrightarrow & Z \end{array}$$

is n-truncated. Thus, ${}^{\leq n}Z \in {}^{\leq n}PreStk$ is n-truncated.

We summarize some other results about schemes.

Proposition 6.4.14. *Any scheme $Z \in Sch$ is a convergent prestack. Also given a convergent prestack Z such that for each n one has ${}^{\leq n}Z \in {}^{\leq n}Sch$, then $Z \in Sch$.*

For the moment we omit the discussion of finiteness conditions.
We just make the following definition that will be useful later.

Definition 6.4.15. A morphism of prestacks $f : \mathscr{X} \to \mathscr{Y}$ is said to be *schematic* if for every $S \in (\mathrm{Sch}^{\mathrm{aff}})_{\mathscr{Y}}$ the pullback $\mathscr{X} \underset{\mathscr{Y}}{\times} S$ is a scheme.

One can easily check the following

Lemma 6.4.16. *Given a schematic morphism of prestacks $\mathscr{X} \to \mathscr{Y}$, then $\mathscr{Y}$ is a scheme then $\mathscr{X}$ is a scheme.*

Chapter 7

Cotangent complex

7.1 Affine theory

7.1.1 Kähler differentials

Given a map of discrete commutative rings $A \to B$ one can define the B-module $\Omega_{B/A}$ of *relative Kähler differentials* given in formula by considering the free B-module generated by the symbols db for $b \in B$ subject to the following relations:

- $d(b + b') = db + db'$;
- $d(bb') = bdb' + b'db$;
- $da = 0$ for all $a \in A$.

This B-module solves the following universal property: given any B-module M and an *A-linear derivation* $d_M : B \to M$, i.e. d_M is A-linear and satisfies

$$d_M(bb') = d_M(b)b' + bd_M(b'), \tag{7.1}$$

there exists an unique factorization

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B/A} \\ & \overset{d_M}{\searrow} & \downarrow{\scriptstyle \varphi_M} \\ & & M \end{array}$$

where φ_B is a B-module map. In other words one has an equivalence of sets

$$\mathrm{Der}_A(B, M) \simeq \mathrm{Hom}_{\mathrm{Mod}_B}(\Omega_{B/A}, M)$$

where the left-hand side denotes the set of A-linear derivations from B to M.

To define a similar object for derived rings we are faced with the problem that we can't write the Leibniz equation (7.1). The following observation comes to the rescue though.

Lemma 7.1.1. *For any B-module M one has equivalences of sets:*

$$Der_A(B, M) \simeq Hom_{Mod_B}(\Omega_{B/A}, M) \simeq Hom_{CAlg^{\heartsuit}_{A/-}}(B, B \oplus M),$$

where $B \oplus M \to B$ is a map of A-algebras and $B \oplus M$ is the square-zero extension *of B by M, i.e. the A-algebra whose underlying A-module is $B \oplus M$ and algebra structure is given by the following formula*

$$(B \oplus M) \times (B \oplus M) \to (B \oplus M) \tag{7.2}$$

$$((b, m), (b'm')) \mapsto (b + b', b'm + bm') \tag{7.3}$$

Actually, the above Lemma didn't quite solve our problem, since we now need to get around writing equation (7.2). However, we have yet another observation to help us.

Lemma 7.1.2. *The pair of functors*

$$\sigma_B : Mod_B \to Ab(CAlg^{\heartsuit}_{/B}) \qquad and \alpha_B : Ab(CAlg^{\heartsuit}_{/B}) \to Mod_B$$
$$M \mapsto (B \oplus M) \stackrel{\pi_B}{\to} B \qquad (B' \stackrel{a}{\to} B) \mapsto \ker a,$$

where $Ab(CAlg^{\heartsuit}_{/B})$ denotes the category of abelian objects in $CAlg^{\heartsuit}_{/B}$ Reference to this notion..

Thus, Lemma 7.1.2 describes the construction $M \to (B \oplus M)$ as a functor realizing an interesting equivalence of categories. The strategy to bypass writing equations in our context is then to generalize this equivalence to the set up of derived rings and their modules.

Remark 7.1.3. In fact we will be able to perform the constructions

$$B \in \mathrm{CAlg}^{\heartsuit} \rightsquigarrow \mathrm{CAlg}^{\heartsuit}_{/B} \rightsquigarrow \mathrm{Ab}(\mathrm{CAlg}^{\heartsuit}_{/B})$$

in great generality, with the property that the resulting category is always abelian, or more generally stable. In fact, this is essentially the defining property of the second squiggly arrow, the abelianization (or stabilization) of the category $\mathrm{CAlg}^{\heartsuit}_{/B}$. In [17, Chapter 6, §1] there is a simple proof that the stabilization of the category of $\mathscr{P}$-algebras in a symmetric monoidal category $\mathscr{C}$ identifies with $\mathscr{C}$, where $\mathscr{P}$ is any (!?) operad. Maybe I can explain how that result goes?

We start by stating a precise result about a version of this result for non-connective algebras.

Proposition 7.1.4. *Let $A \in CAlg^{\mathrm{nc}}$ be an arbitrary[1] derived ring, then one has an of categories*

$$Spctr(CAlg^{\mathrm{nc}}_{/A}) \simeq Mod_A,$$

*where $Spctr(CAlg^{\mathrm{nc}}_{/A})$ means the category of spectrum objects in $CAlg^{\mathrm{nc}}_{/A}$ (see Definition **??**). In particular, one obtains*

$$\Phi_A : ComMonoid(CAlg_{A/-/A}) \stackrel{\simeq}{\to} Mod^{\leq 0}_A. \tag{7.4}$$

Proof. The first statement is [32].

For the second see [32]. □

Notation 7.1.5. We will let

$$\mathrm{SplitSqZ} : \mathrm{QCoh}(S) \to \mathrm{Sch}^{\mathrm{aff}}_{S/}$$
$$\mathscr{F} \mapsto S_{\mathscr{F}} := \mathrm{Spec}(A \oplus \Gamma(S, \mathscr{F})),$$

where $S = \mathrm{Spec}(A)$, denote the functor determined by the inverse of the equivalence (7.4). See [32, Theorem 7.3.4.13] or [17, Chapter 6, Proposition 1.8.3] for more details.

Remark 7.1.6. The following is a consequence of the proof of Proposition 7.1.4. Given $M \in \mathrm{Mod}_A$ and let $\sigma_A(M)$ denote the A-module $M \oplus A$ endowed with an algebra structure $m : \sigma_A(M) \otimes \sigma_A(M) \to \sigma_A(M)$, one has

- the restriction of m to $A \otimes A$ is homotopic to the algebra structure of A;
- the restriction of m to $A \otimes M$ (or $M \otimes A$) is homotopic to the A-module structure of M;
- the restriction of m to $M \otimes M$ is nullhomotopic, i.e. homotopic to the zero map.

In particular, in cohomology one has an algebra structure on

$$H^*(\sigma_A(M)) \simeq H^*(A) \oplus H^*(M)$$

which agrees with the split-square zero extension[2] of $H^*(A)$ by the $H^*(A)$-module $H^*(M)$.

Define $\mathbb{L}_{B/A}$ and its properties.

[1]I.e. Not necessarily connective.
[2]I.e. the algebra structure by writing the equation (7.2) for the cohomology classes.

7.2 Global theory

Let $\mathscr{X}$ be a prestack and consider a point $x : S \to \mathscr{X}$, we let

$$\begin{aligned} \mathrm{Lift}_x : \mathrm{QCoh}(S)^{\leq 0} \to & \mathrm{Spc} \\ \mathscr{F} \mapsto & \mathrm{Maps}_{S/}(S_{\mathscr{F}}, \mathscr{X}) \end{aligned}$$

denote the functor that take $\mathscr{F}$ to the space of lifts of x to the split-square zero extension, i.e. morphisms $\tilde{x} : S_{\mathscr{F}} \to \mathscr{X}$ such that the following diagram

$$\begin{array}{ccc} S_{\mathscr{F}} & & \\ \uparrow & \overset{\tilde{x}}{\searrow} & \\ S & \xrightarrow{x} & \mathscr{X} \end{array}$$

commutes.

Given a morphism $\mathscr{F}_1 \to \mathscr{F}_2$ in $\mathrm{QCoh}(S)^{\leq 0}$, whose induced map on H^0 is surjective, we have[3]

$$\mathscr{F} := 0 \underset{\mathscr{F}_2}{\times} \mathscr{F}_1 \in \mathrm{QCoh}(S)^{\leq 0}.$$

Since the functor

$$\begin{aligned} \mathrm{SplitSqZ} : \mathrm{QCoh}(S)^{\leq 0} &\to \mathrm{Sch}^{\mathrm{aff}}_{S/} \\ \mathscr{F} &\mapsto S_{\mathscr{F}} \end{aligned}$$

sends pullbacks to pushouts, we have

$$S_{\mathscr{F}} \simeq S \underset{S_{\mathscr{F}_2}}{\sqcup} S_{\mathscr{F}_1}.$$

Definition 7.2.1. A prestack $\mathscr{X}$ admits a *pro-cotangent space* at a point $x : S \to \mathscr{X}$ if for every pair as in **??** the morphism

$$\mathrm{Maps}_{S/}(S_{\mathscr{F}}, \mathscr{X}) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathscr{F}_2}, \mathscr{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathscr{F}_1}, \mathscr{X}) \tag{7.5}$$

is an equivalence.

The isomorphism (7.5) implies that we can extend the functor Lift to $\mathrm{QCoh}(S)^{-} := \bigcup_{n \geq 0} \mathrm{QCoh}(S)^{\leq 0}$ by

$$\begin{aligned} \mathrm{Lift}^{-} : \mathrm{QCoh}(S)^{-} &\to \mathrm{Spc} \\ & \mathscr{F} \Omega^k \, \mathrm{Maps}_{S/}(S_{\mathscr{F}[k]}, \mathscr{X}) \Big), \end{aligned}$$

where $k \geq 0$ is any integer such that $\mathscr{F}[k] \in \mathrm{QCoh}(S)^{\leq 0}$. Moreover, we notice that Lift^{-} is an exact functor, since $\mathrm{QCoh}(S)^{-}$ is stable any finite limit or colimit can be written as the limit of sheaves used in the condition (7.5). Thus, there exists an object $T^*_x \mathscr{X} \in \mathrm{Pro}(\mathrm{QCoh}(S)^{-})$ co-representing the functor Lift^{-}[4], which we call the *pro-cotangent space of* $\mathscr{X}$ *at* x.

Notice that for any pullback diagram

$$\begin{array}{ccc} \mathscr{F}'_1 & \longrightarrow & \mathscr{F}_1 \\ \downarrow & & \downarrow \\ \mathscr{F}'_2 & \longrightarrow & \mathscr{F}_2 \end{array}$$

[3]Notice that in general this fiber product would only be (-1)-connective.

[4]Indeed, one definition of $\mathrm{Pro}(\mathrm{QCoh}(S)^{-})$ is as the subcategory of $\mathrm{Fun}(\mathrm{QCoh}(S)^{-}, \mathrm{Spc})$ consisting of left-exact functors.

in the category $\mathrm{QCoh}(S)^{\leq 0}$ one has

$$\mathrm{Maps}_{S/}(S_{\mathscr{F}_1'}, \mathscr{X}) \xrightarrow{\simeq} \mathrm{Maps}_{S/}(S_{\mathscr{F}_2'}, \mathscr{X}) \underset{\mathrm{Maps}_{S/}(S_{\mathscr{F}_2}, \mathscr{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathscr{F}_1}, \mathscr{X}).$$

Indeed, this follows from the fact that

$$\mathrm{Maps}_{S/}(S_{(-)}, \mathscr{X}) \simeq \mathrm{Hom}_{\mathrm{Pro}(\mathrm{QCoh}(S)^-)}(T^*_x\mathscr{X}, -)$$

commutes with finite limits, since the right-side can be computed as a filtered colimit of co-representable functors.

Split Square-zero extensions Before giving the definition of a pro-cotangent complex we need to recall a property from the construction of square-zero extensions.

Let $f : S_1 \to S_2$ be a map of affine schemes, then one has a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(S_1)^{\leq 0} & \xrightarrow{f_*} & \mathrm{QCoh}(S_2)^{\leq 0} \\ \big\downarrow \mathrm{SplitSqZ} & & \big\downarrow \\ \mathrm{Sch}^{\mathrm{aff}}_{S_1/} & \xrightarrow{(-)\underset{S_1}{\sqcup} S_2} & \mathrm{Sch}^{\mathrm{aff}}_{S_2/} \end{array}$$

Thus, given a point $x_2 : S_2 \to \mathscr{X}$, let $x_1 := x_2 \circ f : S_1 \to \mathscr{X}$ there is a map

$$\mathrm{Maps}_{S_2/}((S_2)_{f_*\mathscr{F}}, \mathscr{X}) \to \mathrm{Maps}_{S_1/}((S_1)_{\mathscr{F}}, \mathscr{X}), \tag{7.6}$$

which induces a morphism

$$T^*_{x_1}\mathscr{X} \to f^*T^*_{x_2}\mathscr{X}. \tag{7.7}$$

Definition 7.2.2. We say that $\mathscr{X}$ admits a *pro-cotangent complex* if for every point $(S_2 \xrightarrow{x_2} \mathscr{X}) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathscr{X}}$ and every morphism $f : S_1 \to S_2$ of affine schemes the map (7.7) (equivalently, (2.9)) is an isomorphism.

Definition 7.2.3. If $\mathscr{X}$ admits cotangent spaces and pro-cotangent complex, then we say that $\mathscr{X}$ admits *cotangent complex*. In this case we have an object

$$T^*\mathscr{X} \in \mathrm{QCoh}(\mathscr{X})^-$$

which we call the *cotangent complex of* $\mathscr{X}$.

Relative situation In many cases it is useful to have relative versions of the notions of (pro-)cotangent spaces and (pro-)cotangent complex.

Definition 7.2.4. Given a morphism of prestacks $f : \mathscr{X} \to \mathscr{Y}$ we say that

a) f *admits pro-cotangent spaces* if for every point $x : S \to \mathscr{X}$ the functor

$$\begin{aligned} \mathrm{Lift}_x(\mathscr{X}/\mathscr{Y}) : \mathrm{QCoh}(S)^{\leq 0} &\to \mathrm{Spc} \\ \mathscr{F} &\mapsto \mathrm{Maps}_{S/}(S_{\mathscr{F}}, \mathscr{X}) \underset{\mathrm{Maps}_{S/}(S_{\mathscr{F}}, \mathscr{Y})}{\times} \mathrm{pt} \end{aligned}$$

is pro-representable;

b) f *admits pro-cotangent complex* if the pro-cotangent spaces are functorial as in Definition 7.2.2. In this case we denote by $T^*_x(\mathscr{X}/\mathscr{Y}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$ the co-representing object.

Remark 7.2.5. In the case where $\mathscr{X}$ and $\mathscr{Y}$ also admit pro-cotangent complexes for every $x : S \to \mathscr{X}$ and $y := f \circ x$ one has an exact sequence

$$T^*_y \mathscr{Y} \to T^*_x \mathscr{X} \to T^*_x(\mathscr{X}/\mathscr{Y}).$$

Moreover, these induce the following exact sequence

$$f^* T^* \mathscr{Y} \to T^* \mathscr{X} \to T^*(\mathscr{X}/\mathscr{Y})$$

in the category $\mathrm{Pro}(\mathrm{QCoh}(\mathscr{X})^-)$.

Notation 7.2.6. When considering affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ it is normal to denote

$$T^*(\operatorname{Spec} B/\operatorname{Spec} A) = \mathbb{L}_{B/A}.$$

Notice that some references (e.g. [35, Chapter ?]) use $\mathbf{L}_{\mathscr{X}/\mathscr{Y}}$ for what we denoted by $T^*(\mathscr{X}/\mathscr{Y})$.

Remark 7.2.7. One can ask to compare the cotangent complex as defined above with previous objects in the literature that have the same name. We claim that for $A \to B$ a morphism of discrete commutative algebras, one has a canonical identification

$$\tau^{\geq -1} \mathbb{L}_{A/B} \simeq NL_{A/B},$$

where $NL_{A/B}$ is the *naive cotangent complex* associated to $A \to B$ (see [Stacks, Tag 00S0]).

Exercise 7.2.8. Given a morphism of prestacks $f : \mathscr{X} \to \mathscr{Y}$ prove that f admits a cotangent spaces if and only if for every affine scheme $S \to \mathscr{Y}$ the pullback $\mathscr{X} \underset{\mathscr{Y}}{\times} S$ admits cotangent spaces.

Example 7.2.9. Given a commutative algebra A and $V \in \mathrm{Mod}_A$ consider $B := \mathrm{Sym}_A(V)$, i.e. the free A-algebra. Recall that $\mathrm{Sym}_A(-) : \mathrm{Mod}_A \to \mathrm{CAlg}_{A/}$ is left adjoint to $\mathrm{oblv} : \mathrm{CAlg}_{A/} \to \mathrm{Mod}_A$. We want to compute $T^*(\mathrm{Spec}(B)/\mathrm{Spec}(A))$. Given a commutative algebra R and $M \in \mathrm{Mod}_R^{\leq 0}$ consider $x : \operatorname{Spec} R \to \operatorname{Spec} B$ and a lift $\tilde{x} : \mathrm{Spec}(R \oplus M) \to \operatorname{Spec} A$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow{\scriptstyle \tilde{x}} & \swarrow & \big\downarrow{\scriptstyle x} \\ R & \longrightarrow & R \oplus M \end{array}$$

We are interested in the space of dashed arrows filling the diagram above, which by definition is given by

$$\mathrm{Hom}_{\mathrm{Mod}_R}(T^*(\mathrm{Spec}(B)/\mathrm{Spec}(A)), M) \simeq \mathrm{Maps}_{S/}(S_M, \operatorname{Spec} B) \underset{\mathrm{Maps}_{S/}(S_M, \operatorname{Spec} A)}{\times} \mathrm{pt} \tag{7.8}$$

where $S = \operatorname{Spec} R$ and $S_M = \mathrm{Spec}(R \oplus M)$. Thus, we compute:

$$\mathrm{Maps}_{S/}(S_M, \operatorname{Spec} B) \simeq \mathrm{Hom}_{\mathrm{CAlg}_{/R}}(B, R \oplus M)$$
$$\mathrm{Maps}_{S/}(S_M, \operatorname{Spec} A) \simeq \mathrm{Hom}_{\mathrm{CAlg}_{/R}}(A, R \oplus M)$$

Consider the following diagram where each row and column is a fiber sequence

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathrm{Mod}_R}(T^*(\mathrm{Spec}(B)/\mathrm{Spec}(A)), M) & \longrightarrow & \mathrm{Maps}_{S/}(S_M, \operatorname{Spec} B) & \longrightarrow & \mathrm{Maps}_{S/}(S_M, \operatorname{Spec} A) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ F_1 & \longrightarrow & \mathrm{Hom}(B, R \oplus M) & \longrightarrow & \mathrm{Hom}(A, R \oplus M) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ F_2 & \longrightarrow & \mathrm{Hom}(B, R) & \longrightarrow & \mathrm{Hom}(A, R) \end{array}$$

we claim that

$$F_1 \simeq \mathrm{Hom}_{\mathrm{CAlg}_{A/}}(B, R \oplus M) \quad \text{and} \quad F_2 \simeq \mathrm{Hom}_{\mathrm{CAlg}_{A/}}(B, R),$$

thus, one has that

$$\mathrm{Hom}_{\mathrm{Mod}_R}(T^*(\mathrm{Spec}(B)/\mathrm{Spec}(A)), M) \simeq \mathrm{Fib}\left(\mathrm{Hom}_{\mathrm{CAlg}_{A/}}(B, R \oplus M) \to \mathrm{Hom}_{\mathrm{CAlg}_{A/}}(B, R)\right),$$

by applying the free-forget adjunction from above we have

$$\begin{aligned}\mathrm{Fib}\left(\mathrm{Hom}_{\mathrm{CAlg}_{A/}}(B, R \oplus M) \to \mathrm{Hom}_{\mathrm{CAlg}_{A/}}(B, R)\right) &\simeq \mathrm{Fib}\left(\mathrm{Hom}_{\mathrm{Mod}_A}(V, R \oplus M) \to \mathrm{Hom}_{\mathrm{Mod}_A}(V, R)\right) \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_A}(V, \mathrm{Fib}(R \oplus M \to R)) \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_A}(V, M) \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_B}(V \otimes_A B, M)\end{aligned}$$

Thus, one obtains that $T^*(\mathrm{Spec}\, B/\mathrm{Spec}\, A) \simeq V \otimes_A B$.

Similarly to what happens for the module of Kähler differentials the cotangent complex on affine schemes has the following properties:

Proposition 7.2.10. *(a) Given a sequence of morphisms of commutative algebras $A \to B \to C$, one obtains a fiber/cofiber sequence:*

$$\mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$$

in the category Mod_C.

(b) Given a pullback square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

in the category CAlg, the canonical map

$$\mathbb{L}_{B/A} \otimes_B B' \to \mathbb{L}_{B'/A'}$$

is an isomorphism.

Proof. Say something about how to obtain this result. □

Example 7.2.11. For any $n \geq 0$, consider the A-algebra $A \oplus A[n]$ endowed with the square-zero extension structure. We want to compute $\mathbb{L}_{A \oplus A[n]/A}$. There are two cases: *Case (i)*: n is odd, then $A \oplus A[n] \simeq \mathrm{Sym}_A(A[n])$ What is a formal way to justify this?, so the computation of Example 7.2.9 gives

$$\mathbb{L}_{A \oplus A[n]/A} \simeq (A \oplus A[n]) \otimes_A A[n] \simeq A[n] \oplus A[2n].$$

Case (ii): $n \geq 2$ is even. Notice that for $n \geq 1$ the following

$$\begin{array}{ccc} A \oplus A[n-1] & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus A[n] \end{array}$$

is a pullback diagram of commutative algebras. Thus, Proposition 7.2.10 (ii) implies that

$$\mathbb{L}_{A \oplus A[n]/A} \simeq \mathbb{L}_{A/A \oplus A[n-1]} \otimes_A (A \oplus A[n]).$$

So we are reduced to calculating $\mathbb{L}_{A/A\oplus A[n-1]}$. Consider the morphisms $A \to A \oplus A[n-1] \to A$, then one has a fiber-cofiber sequence

$$A \otimes_{A\oplus A[n-1]} \mathbb{L}_{A\oplus A[n-1]/A} \to \mathbb{L}_{A/A} \to \mathbb{L}_{A/A\oplus A[n-1]}. \tag{7.9}$$

Since the middle term vanishes one obtains:

$$\mathbb{L}_{A/A[n-1]} \simeq A \otimes_{A\oplus A[n-1]} \mathbb{L}_{A\oplus A[n-1]/A}[1] \simeq (A \otimes_{A\oplus A[n-1]} A[n-1] \oplus A[2n-2])[1] \simeq A[n-1][1] \simeq A[n]$$

Thus, one obtains

$$\mathbb{L}_{A\oplus A[n]/A} \simeq A[n] \otimes_A (A \oplus A[n]) \simeq A[n] \oplus A[2n].$$

Case (iii): $n = 0$. In this case the result follows from Remark 7.2.7 that one has

$$H^0\mathbb{L}_{A\oplus A/A} \simeq \Omega_{A\oplus A/A} \simeq A$$

where A is discrete.

Finally, we have the following result called connectivity estimate, which is a very useful tool for computing the cotangent complex or proving properties about it.

Theorem 7.2.12. *Given a morphism $f : A \to B$ in CAlg whose fiber is n-connective for some $n \geq 0$, then there exists a $2n$-connective morphism*

$$\epsilon_f : B \otimes_A \mathit{Cofib}(f) \to \mathbb{L}_{B/A}.$$

Proof. See [32, Theorem 7.4.3.1] for a proof. □

Here is a consequence of the above result.

Corollary 7.2.13. *Let $f : A \to B$ be a morphism of commutative algebras.*

(a) If f is n-connective, i.e. $\mathit{Cofib}(f) \in \mathit{Mod}_B^{\leq -n}$, then $\mathbb{L}_{B/A}$ is n-connective.

(b) Assume that f induces an equivalence ${}^{\mathrm{c}\ell}f : H^0\,A \to H^0\,B$ and that $\mathbb{L}_{B/A}$ is n-connective, then $\mathit{Cofib}(f)$ is n-connective.

Proof. We prove (a) and leave (b) as an exercise.

Consider the fiber/cofiber sequence

$$\mathit{Fib}(\epsilon_f) \to B \otimes_A \mathrm{Cofib}(f) \to \mathbb{L}_{B/A},$$

then $B \otimes_A \mathrm{Cofib}(f) \in \mathrm{Mod}_B^{\leq -n}$ and $\mathit{Fib}(\epsilon_f) \in \mathrm{Mod}_B^{\leq -2n} \subset \mathrm{Mod}_B^{\leq -n+1}$ implies that $\mathbb{L}_{B/A} \in \mathrm{Mod}_B^{\leq -n}$. □

Corollary 7.2.14. *For any $S \in \mathit{Sch}^{\mathrm{aff}}$ one has $T^*S \in \mathit{QCoh}(S)^{\leq 0}$.*

Corollary 7.2.15. *Given a morphism $f : S \to T$ in $\mathit{Sch}^{\mathrm{aff}}$, then f is an isomorphism if and only if the following hold:*

1) ${}^{\mathrm{c}\ell}f : {}^{\mathrm{c}\ell}S \to {}^{\mathrm{c}\ell}T$ is an isomorphism;

2) $T^*(T/S)$ vanishes.

7.3 Square-zero extensions

Motivation: let's discuss the notion of square-zero extensions for classical affine schemes and their relation to the cotangent space, i.e. module of Kähler differentials.

Given a discrete commutative algebra $R \in \mathrm{CAlg}^{\mathrm{disc.}}$ a *square-zero extension* is a discrete commutative algebra R' with a surjective morphism

$$\varphi : R' \to R$$

such that $I^2 = 0$, where $I := \ker \varphi$.

Notice that for any $M \in \mathrm{Mod}_R^{\heartsuit}$, up to isomorphism there is an unique discrete commutative algebra structure on $R \oplus M$, such that the canonical projection

$$\varphi_M : R \oplus M \to R$$

makes $R \oplus M$ into a square-zero extension of R with $\ker \varphi_M = M$. Moreover, one has a splitting, i.e. a morphism $s : R \to R \oplus M$ such that $\varphi_M \circ s = \mathrm{id}_R$.

Given $\varphi : R' \to R$ a square-zero extension of R by a module $M \in \mathrm{Mod}_R^{\heartsuit}$, i.e. $\ker \varphi = M$, then one can consider the morphism of discrete commutative algebras[5]

$$(R \oplus M) \underset{R}{\times} R' \to R'$$
$$(r, m, r') \mapsto r \cdot r' + m$$

One has the following result:

Lemma 7.3.1. *For any square-zero extension $\varphi : R' \to R$, one has (canonical?) bijections:*

$$Aut_{CAlg^{\mathrm{disc.}}_{/R}}(R') \simeq Der(R, G) \simeq Hom_{Mod_R^{\heartsuit}}(\Omega_R, M).$$

Proof. Provide some argument. □

Assume that T^*S is perfect, one can check that:

$$H^{-i}(T^*S) \simeq \mathrm{Ext}^i(T^*S, \mathscr{O}_S)^{\vee}. \tag{7.10}$$

Thus, when S is classical of finite type one has $H^{-1}(T^*S) \simeq 0$ for $i \geq 1$. And we obtain that

$$H^0(T^*S) \simeq \Omega_R,$$

where $S = \operatorname{Spec} R$ for some discrete commutative algebra R.

So we can add the following set to the bijections in Lemma 7.3.1:

$$\mathrm{Aut}_{\mathrm{CAlg}^{\mathrm{disc.}}_{/R}}(R') \simeq \mathrm{Hom}_{\mathrm{Mod}_R^{\heartsuit}}(H^0 T^*S, M).$$

Now the equation (7.10) makes one wonder if there is a way to describe $H^{-i}(T^*S)$ in terms of extensions of S for $i \geq 1$.

Before answering this question we need to introduce some notation.

For $S \in \mathrm{Sch}^{\mathrm{aff}}$ consider the category:

$$\mathrm{SqZ}(S) := (\mathrm{QCoh}(S)^{\leq -1}_{T^*S/})^{\mathrm{op}}.$$

Notice that given $\gamma : T^*S \to \mathscr{F}$ a morphism in $\mathrm{QCoh}(S)^{\leq -1}$ one has

$$S_{\mathscr{F}} \xrightarrow{\gamma^*} S_{T^*S}$$

in the category $\mathrm{Sch}^{\mathrm{aff}}_{S//S}$. We also let

$$S_{T^*S} \xrightarrow{\partial} S$$

[5]Here we abuse notation and simply write m for the image of $M \simeq \ker \varphi \hookrightarrow R'$.

denote the map induced by $\mathrm{id}_{T^*S} \in \mathrm{Hom}_{\mathrm{QCoh}(S)}(T^*S, T^*S)$[6].

Definition 7.3.2. Rephrase this section so that the equation below is in here.

Thus, we define

$$\mathrm{SqZ}(T^*S \xrightarrow{\gamma} \mathscr{F}) := S \underset{\sqcup}{S_{\mathscr{F}}} S,$$

where the left morphism is the canonical projection and the right morphism is the composite

$$S_{\mathscr{F}} \xrightarrow{\gamma^*} S_{T^*S} \xrightarrow{\partial} S.$$

Here is one way to think about the functor SqZ. Let

$$\mathrm{Sch}^{\mathrm{aff}}_{S/,\mathrm{inf-closed}} := \left\{ f : S \to T \mid df^* : f^*T^*T \to T^*S \ \text{ st } \ H^0(d^*f) \text{ is surjective } \right\}.$$

Notice that the condition on df^* is equivalent to requiring that the relative cotangent complex is 1-connective, i.e. $T^*(S/T) \in \mathrm{QCoh}(S)^{\leq -1}$. The following is a consequence of the definitions

Lemma 7.3.3. *There is a pair of adjoint functor*

$$SqZ : (QCoh(S)^{\leq -1}_{T^*S/})^{\mathrm{op}} \rightleftarrows Sch^{\mathrm{aff}}_{S/,\mathrm{inf-closed}} : Q,$$

where

$$Q(S \xrightarrow{f} T) := T^*S \to T^*(S/T).$$

Warning 7.3.4. The functor $\mathrm{SqZ} : (\mathrm{QCoh}(S)^{\leq -1}_{T^*S/})^{\mathrm{op}} \to \mathrm{Sch}^{\mathrm{aff}}_{S/}$ is *not* fully faithful. Give an example? The point being that in derived algebraic geometry, being a square-zero extension is not just a property but extra data, i.e. the equation $I^2 = 0$ when considered homotopically involves higher coherences that are not included in the map $f : S \to T$.

We remedy the warning with the notion of an n-small extension.

Definition 7.3.5. A morphism $\mathrm{Spec}\, B = S \xrightarrow{f} T = \mathrm{Spec}\, A$, whose corresponding morphism of commutative algebras we denote $\varphi : A \to B$, is an *n-small extension* if

a) $\mathrm{Fib}\, \varphi \in \mathrm{Vect}^{\leq -2n}$

b) $\mathrm{Fib}\, \varphi \otimes_A \mathrm{Fib}\, \varphi \to \mathrm{Fib}\, \varphi$ is homotopic to zero.

Rewrite the above conditions geometrically.

The following is [32, Theorem 7.4.1.23]:

Proposition 7.3.6. *One has an equivalence of categories:*

$$\left\{ S \xrightarrow{f} T \mid f \text{ is an } n\text{-small extension} \right\} \simeq (QCoh(S)^{\geq -2n, \leq -1}_{T^*S/-})^{\mathrm{op}}.$$

In which category is T^*S? Does it matter?

We will address the failure of fully faithfulness by encoding this data slightly differently.

[6] Notice that the element $\mathrm{id}_{T^*S} \in \mathrm{Hom}_{\mathrm{QCoh}(S)}(T^*S, T^*S)$ gives rise to a lift of the identity morphism of S:

$$\begin{array}{ccc} S_{T^*S} & & \\ \uparrow & \searrow^{\partial} & \\ S & \xrightarrow{\mathrm{id}_S} & S. \end{array}$$

Definition 7.3.7. For a fixed $\mathscr{I} \in \mathrm{QCoh}(S)$, let $\mathscr{F} := \mathscr{I}[1]$ we will call

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(T^*S, \mathscr{F})$$

the space of *square-zero extensions of S by $\mathscr{I}$*.

Here is a justification for the terminology. Consider

$$S \overset{\imath}{\hookrightarrow} T = \mathrm{SqZ}(T^*S \to \mathscr{I}[1]),$$

then one has an exact sequence:

$$\imath_* \mathscr{I} \to \mathscr{O}_T \to \imath_* \mathscr{O}_S,$$

thus we can say that $\mathscr{I}$ plays the role of the "ideal of definition" of S inside T.

Notice that the following diagram commutes:

$$\begin{array}{ccc} (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} & \xrightarrow{\mathscr{F} \mapsto (T^*S \xrightarrow{0} \mathscr{F})} & (\mathrm{QCoh}(S)^{\leq -1}_{T^*S/})^{\mathrm{op}} \\ \downarrow \mathrm{SplitSqZ} & & \downarrow \mathrm{SqZ} \\ \mathrm{Sch}^{\mathrm{aff}}_{S//S} & \longrightarrow & \mathrm{Sch}^{\mathrm{aff}}_{S/} \end{array}$$

where $\mathrm{SplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to \mathrm{Sch}^{\mathrm{aff}}_{S//S}$ sends $\mathscr{F}$ to

$$S \to S_{\mathscr{F}} \to S.$$

Indeed, we notice that since $S_{\mathscr{F}} \to S$ is a closed nil-isomorphism, i.e. ${}^{\mathrm{red}}S_{\mathscr{F}} \xrightarrow{\simeq} {}^{\mathrm{red}}S$, and ${}^{\mathrm{c}\ell}S_{\mathscr{F}} \hookrightarrow {}^{\mathrm{c}\ell}S$, the pushout

$$S \underset{S_{\mathscr{F}[1]}}{\sqcup} S$$

can be performed in the category of affine schemes. So we have

$$S \underset{S_{\mathscr{F}[1]}}{\sqcup} S \simeq \mathrm{Spec}(R \underset{R \oplus M[1]}{\times} R) \simeq \mathrm{Spec}(R \oplus (0 \underset{M[1]}{\times} 0)) \simeq \mathrm{Spec}(R \oplus M) = S_{\mathscr{F}},$$

where $M = \Gamma(S, \mathscr{F})$ and $S = \mathrm{Spec}\, R$.

Theorem 7.3.8. *(a) For $S \in {}^{\mathrm{c}\ell}Sch^{\mathrm{aff}}$ one has an equivalence of categories*[7]

$$(QCoh(S_0)^{\heartsuit}[1]_{T^*S_0/})^{\mathrm{op}} \simeq \left\{ S_0 \overset{\imath}{\hookrightarrow} S_0' \mid \text{ classical square-zero extensions } \right\}.$$

(b) For $S_n \in {}^{\leq n}Sch^{\mathrm{aff}}$ one has an equivalence of categories:

$$(QCoh(S_n)^{\heartsuit}[1]_{T^*S_n/})^{\mathrm{op}} \simeq \left\{ (S_n \xrightarrow{\imath_n} S_{n+1}) \in {}^{\leq n}Sch^{\mathrm{aff}} \mid {}^{\leq n}S_n \xrightarrow{\simeq} {}^{\leq n}S_{n+1} \right\}.$$

Proof. Consider $\imath_n : S_n \hookrightarrow S_{n+1}$ whose truncation ${}^{\leq n}\imath_n$ is an equivalence. This gives an exact sequence:

$$\imath_{n,*}\mathscr{F}[-1] \to \mathscr{O}_{S_{n+1}} \to \imath_{n,*}\mathscr{O}_{S_n}$$

where $\mathscr{F} \in \mathrm{QCoh}(S_n)^{\heartsuit}[n+2]$.

We claim that there exists a morphism $\gamma : T^*(S_n) \to \mathscr{F}$ such that

$$S_{n+1} \simeq S_n \underset{(S_n)_{\mathscr{F}}}{\sqcup} S.$$

[7] By a classical square-zero extension we mean that $\imath$ is a closed embedding of classical affine schemes, whose ideal of definition squares to 0.

Consider the exact sequence:

$$\imath^* T^*(S_{n+1}) \to T^*(S_n) \to T^*(S_n/S_{n+1})$$

we claim that

$$H^k(T^*(S_n/S_{n+1})) = \begin{cases} 0, & \text{for } k \geq -n-1; \mathscr{F}, & \text{for } k = -n-2. \end{cases}$$

These are consequences of the connectivity estimates. Indeed, when $k = 0$ let $S' = \operatorname{Spec} R'$ and $S = \operatorname{Spec} R$, then the induced map

$$\varphi : R' \to R$$

has $\operatorname{Cofib}\varphi$ 1-connective, i.e. $\operatorname{Cofib}\varphi \in \operatorname{Mod}_R^{\leq -1}$, so by Theorem 7.2.12 the map

$$R' \otimes_R \operatorname{Cofib}\varphi \to \mathbb{L}_{R'/R}$$

is 2-connective, i.e.

$$\mathscr{I}[1] \to T^*(S'/S)$$

induces an isomorphism on $\operatorname{Mod}_{R'}^{\geq -1, \leq 0}$, which gives

$$H^{-1}(T^*(S'/S)) \simeq H^0(\mathscr{I}) \simeq \mathscr{I}.$$

For $k \geq 1$ we have that

$$\imath_*(\mathscr{F}[-1])[1] \to T^*(S_n/S_{n+1})$$

is an isomorphism on $\operatorname{QCoh}(S_n)^{\geq -2n+1, \leq 0}$. In particular, one obtains

$$H^i(T^*(S_n/S_{n+1})) \simeq 0, \quad \text{for} \quad i \geq -k-1,$$

and

$$H^{-k-2}(T^*(S_n/S_{n+1})) \simeq \mathscr{F}.$$

□

Remark 7.3.9. The above discussion of square-zero extensions makes sense for (derived) schemes as well. Whereas the initial input is that for any scheme $X \in \operatorname{Sch}$, one has

$$T_x^* X \in \operatorname{QCoh}(S)^{\leq 0} \quad \text{for any} \quad (S \xrightarrow{x} X) \in \operatorname{Sch}^{\operatorname{aff}}_{/X},$$

because

$$\begin{array}{ccc}
\operatorname{Maps}_{S/}(S_{\mathscr{F}}, X) & \longrightarrow & \operatorname{Maps}_{S/}(S_{\mathscr{F}_1}, X) \underset{\operatorname{Maps}_{S/}(S_{\mathscr{F}_2}, X)}{\times} \operatorname{pt} \\
\downarrow \simeq & & \downarrow \simeq \\
\operatorname{Hom}_{\operatorname{Sch}}(S_{\mathscr{F}}, X) \underset{\operatorname{Hom}_{\operatorname{Sch}}(S,X)}{\times} \operatorname{pt} & \longrightarrow & \left(\operatorname{Hom}_{\operatorname{Sch}}(S_{\mathscr{F}_2}, X) \underset{\operatorname{Hom}_{\operatorname{Sch}}(S_{\mathscr{F}_1}, X)}{\times} \operatorname{Hom}_{\operatorname{Sch}}(S, X)\right) \underset{\operatorname{Hom}_{\operatorname{Sch}}(S,X)}{\times} \operatorname{pt}
\end{array}$$

and one has

$$S_{\mathscr{F}_2} \underset{S_{\mathscr{F}_1}}{\sqcup} S \simeq S$$

in the category Sch.

Remark 7.3.10. One can rephrase Theorem 7.3.8 above as there is a fully faithful functor

$${}^{\leq n}\operatorname{Sch}^{\operatorname{aff}} \hookrightarrow \operatorname{SqZ}(\operatorname{Sch}^{\operatorname{aff}}) \underset{\operatorname{Sch}^{\operatorname{aff}} \times \operatorname{Sch}^{\operatorname{aff}}}{\times} {}^{\leq n}\operatorname{Sch}^{\operatorname{aff}} \times {}^{\leq n+1}\operatorname{Sch}^{\operatorname{aff}},$$

where $\mathrm{SqZ}(\mathrm{Sch}^{\mathrm{aff}})$ is the category whose objects are

$$(S \in \mathrm{Sch}^{\mathrm{aff}}, (T^*S \xrightarrow{\gamma} \mathscr{F}) \in \mathrm{QCoh}(S)^{\leq -1})$$

and morphisms are the data of f and α as below:

$$f : S \to S' \qquad \begin{array}{ccccc} T^*S' & \longrightarrow & f_*T^*S & \xrightarrow{f_*\gamma} & f_*\mathscr{F} \\ & & \searrow^{\gamma'} & & \downarrow{\scriptstyle \alpha} \\ & & & & \mathscr{F}' \end{array}$$

Chapter 8

Étale and smooth morphisms

8.1 Étale and smooth morphisms

The cotangent complex can also be used to understand étale and smooth morphisms.

We start by recalling how we define these notions for a morphism between schemes.

Definition 8.1.1. (i) A morphism $f : X \to S$ from a scheme X to an affine scheme S is *étale* (resp. *smooth*) if for some (equivalently for any) Zariski cover $\sqcup_I Z_i \simeq Z \to X$ the composites

$$Z_i \to S$$

are *'etale* (resp. *smooth*) morphism of affine schemes, for all $i \in I$.

(ii) A morphism $f : X \to Y$ between objects of Sch is *étale* (resp. *smooth*) if for every affine scheme $S \to Y$ the morphism

$$X \underset{Y}{\times} S \to S$$

is *'etale* (resp. *smooth*).

Remark 8.1.2. If Y has affine diagonal we notice that for any $(S \to Y) \in \mathrm{Sch}^{\mathrm{aff}}_{/Y}$ the fiber product $X \underset{Y}{\times} S$ is affine, hence we don't need case (i) of Definition 8.1.1 since then the definition can be directly generalized from the case of affine schemes.

Remark 8.1.3. Given a morphism $f : X \to Y$ in Sch we claim that f is étale (resp. smooth) if and only if for any étale cover $\sqcup_I Z_i \simeq Z \to X$ all the composites

$$Z_i \to X \to Y$$

are étale (resp. smooth). Indeed, give a proof of this.

We have the following result (cf. Reference to TV statement.).

Proposition 8.1.4. *Let $f : X \to Y$ be a finitely presented morphism between objects of Sch then we have:*

(a) f is étale *if and only if $T^*(X/Y) = 0$;*

(b) f is smooth *if and only if $T^*(X/Y) \in Vect(X)$, i.e. $T^*(X/Y)$ is dualizable in $QCoh(X)^{\leq 0}$.*

Proof. We claim that the result can be reduced to the statement for affine schemes. Indeed, by Remark 8.1.3 we can reduced to the case X is affine, and by definition we can reduced to the case where Y is affine.

Let $\operatorname{Spec} B = X \to Y = \operatorname{Spec} A$ be an étale morphism, i.e. $A \to B$ is flat and $H^0(A) \to H^0(B)$ is étale. Since $A \to B$ is flat the following diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ H^0(A) & \longrightarrow & H^0(B) \end{array}$$

is a push-out, whence by Proposition 7.2.10 we have

$$\mathbb{L}_{B/A} \otimes_B H^0(B) \simeq \mathbb{L}_{H^0(B)/H^0(A)}.$$

By Remark **??** we have that $\mathbb{L}_{H^0(B)/H^0(A)}$ recovers the classically defined $L_{H^0(B)/H^0(A)}$, and by [Stacks, Tag 08R2] $H^0(A) \to H^0(B)$ étale implies that $L_{H^0(B)/H^0(A)}$ vanishes.

Now, by **??**, we have a spectral sequence, whose E_2-page is:

$$H^p(H^0(B) \underset{H^\bullet B}{\otimes} H^q(\mathbb{L}_{B/A})) \Rightarrow H^{p+q}(H^0(B) \underset{B}{\otimes} \mathbb{L}_{B/A}).$$

We notice that for $H^{-1}(\mathbb{L}_{B/A})$ to be non-zero we would need that

$$\mathrm{Tor}^i(H^0(B), H^{-i-1}(\mathbb{L}_{B/A})) \neq 0$$

for some $i \geq 1$. Since $A \to B$ is finitely presented $\mathbb{L}_{B/A}$ is perfect, hence we have $H^{-k}(\mathbb{L}_{B/A}) = 0$ for $k \ll 0$.

We claim that by induction we obtain that $H^k(\mathbb{L}_{B/A}) = 0$. Finish this argument correctly!

Now assume that $\mathbb{L}_{B/A} = 0$, we will use induction on n to prove that

$$\tau^{\geq -n}(A) \to \tau^{\geq -n}(B)$$

is étale for all $n \geq 0$.

Consider the fiber sequences:

$$\begin{array}{ccccc}
\mathbb{L}_{H^0(A)/A} \underset{H^0(A)}{\otimes} H^0(B) & \longrightarrow & \mathbb{L}_{H^0(B)/A} & \longrightarrow & \mathbb{L}_{H^0(B)/H^0(A)} \\
 & & \uparrow & & \\
\mathbb{L}_{B/A} \underset{B}{\otimes} H^0(B) & \longrightarrow & \mathbb{L}_{H^0(B)/A} & \longrightarrow & \mathbb{L}_{H^0(B)/B}.
\end{array}$$

Since $B \xrightarrow{\alpha} H^0(B)$ has $\mathrm{Cofib}\,\alpha \in \mathrm{Mod}_B^{\leq -2}$, Corollary 7.2.13 implies that

$$H^i(\mathbb{L}_{H^0(B)/B}) \simeq H^i(H^0(B) \underset{B}{\otimes} \mathrm{Cofib}\,\alpha)$$

for $i \geq -3$; so $H^i(\mathbb{L}_{H^0(B)/B}) = 0$ for $i \geq -1$.

Since $\mathbb{L}_{B/A} \simeq 0$, one has $\tau^{\geq -1}(\mathbb{L}_{H^0(B)/H^0(A)}) \simeq 0$.

By Remark 7.2.7 $\tau^{\geq -1}(\mathbb{L}_{H^0(B)/H^0(A)})$ is the naive cotangent complex of $H^0(A) \to H^0(B)$, thus the morphism $\tau^{\geq 0}(A) \to \tau^{\geq 0}(B)$ is étale by [Stacks, Definition 10.143.1].

Moreover, one can check that $\tau^{\geq -n}(\mathbb{L}_{\tau^{\geq -n}(B)/\tau^{\geq -n}(A)}) \simeq \tau^{\geq -n}(\mathbb{L}_{B/A}) \simeq 0$ for all $n \geq 0$. So given $M \in \mathrm{Mod}_{H^0(B)}^{\heartsuit}$ and a map $\gamma : \mathbb{L}_{\tau^{\geq -n}(B)/\tau^{\geq -n}(A)} \to M[n+1]$, we can consider the square-zero extension $f : \tau^{\geq -n}(B) \to \tau^{\geq -n}(B) \oplus M[n]$ depicted as:

$$\begin{array}{ccc}
\tau^{\geq -n}(A) & \longrightarrow & \tau^{\geq -n}(B) \oplus M[n] \\
\downarrow & \overset{f}{\nearrow} & \downarrow \\
\tau^{\geq -n}(B) & \longrightarrow & \tau^{\geq -n}(B)
\end{array}$$

However, f is determined by $\tilde{f} : B \to \tau^{\geq -n}(B) \oplus M[n]$ which by Theorem 7.2.12 is determined by a morphism

$$\mathbb{L}_{B/A} \to M[n+1]$$

which is canonically 0, since $\mathbb{L}_{B/A}$.

Thus, one has that $\mathbb{L}_{\tau^{\geq -n}(B)/\tau^{\geq -n}(A)} \in \mathrm{Mod}_{\tau^{\geq -n}(B)}^{\leq -n-1}$.

Now we consider the sequences:

$$\begin{array}{ccccc} \mathbb{L}_{\tau^{\geq -n}(A)} \underset{\tau^{\geq -n}(A)}{\otimes} \tau^{\geq -n-1}(A) & \longrightarrow & \mathbb{L}_{\tau^{\geq -n-1}(A)} & \longrightarrow & \mathbb{L}_{\tau^{\geq -n-1}(A)/\tau^{\geq -n}(A)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{\tau^{\geq -n}(B)} \underset{\tau^{\geq -n}(B)}{\otimes} \tau^{\geq -n-1}(B) & \longrightarrow & \mathbb{L}_{\tau^{\geq -n-1}(B)} & \longrightarrow & \mathbb{L}_{\tau^{\geq -n-1}(B)/\tau^{\geq -n}(B)} \end{array} \tag{8.1}$$

by tensoring the upper row in (8.1) by $(-) \underset{\tau^{\geq -n+1}(A)}{\otimes} \tau^{\geq -n+1}(B)$, the resulting left and middle columns become $(n+1)$-connective, since $\tau^{\geq -n+1}(A) \to \tau^{\geq -n+1}(B)$ is étale. So

$$H^{-n-1}(\mathbb{L}_{\tau^{\geq -n-1}(A)/\tau^{\geq -n}(A)} \underset{\tau^{\geq -n+1}(A)}{\otimes} \tau^{\geq -n+1}(B)) \simeq H^{-n-1}(\mathbb{L}_{\tau^{\geq -n+1}(B)/\tau^{\geq -n}(B)})$$

By Exercise 8.1.5, one obtains:

$$H^{-n}(A) \underset{H^0(A)}{\otimes} H^0(B) \simeq H^{-n}(B).$$

This proves that $A \to B$ is flat, hence finishes checking that $A \to B$ is étale.

Check this proof! □

Exercise 8.1.5. Let $A \in \text{CAlg}$ prove that

$$H^{-n-1}(\mathbb{L}_{\tau^{\geq -n+1}(A)/\tau^{\geq -n}(A)}) \simeq H^{-n}(A).$$

The following result follows from tracing through the definitions and using the connectivity estimates:

Lemma 8.1.6. *Let $\mathscr{X} \in Stk_{\leq 1}$ be a 1-truncated stack and consider the canonical map*[1]

$$\imath : {}^{\text{der}}({}^{\text{c}\ell}\mathscr{X}) \to \mathscr{X}.$$

Then one has comparison maps:

$$\imath^* T^* \mathscr{X} \overset{\varphi'_{\mathscr{X}}}{\to} T^{*\text{der}}({}^{\text{c}\ell}\mathscr{X}) \overset{\varphi''_{\mathscr{X}}}{\to} \mathsf{T}^{*\text{c}\ell}\mathscr{X},$$

where $\mathsf{T}^{*\text{c}\ell}\mathscr{X}$ denotes the cotangent complex of an algebraic stack in the classical sense (see [Stacks, ??]) and

(a) $\varphi'_{\mathscr{X}}$ is 1-connective;

(b) $\varphi''_{\mathscr{X}}$ is 2-connective.

In particular, for any 1-truncated classical stack $\mathscr{X}_0$ one has

$$\tau^{\geq -1}(T^{*\text{der}}\mathscr{X}_0) \overset{\simeq}{\to} \tau^{\geq -1}(\mathsf{T}^*\mathscr{X}_0)$$

Corollary 8.1.7. *For any smooth classical scheme Z one has:*

$$T^{*\text{der}} Z \simeq \mathsf{T}^* Z[0],$$

where $\mathsf{T}^*Z[0]$ denotes the usual cotangent vector bundle of Z placed in degree 0. Give a different notation to this.

[1]Here

$${}^{\text{der}}({}^{\text{c}\ell}\mathscr{X}) := {}^{\text{L}}\text{LKE}_{{}^{\text{c}\ell}\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}}({}^{\text{c}\ell}\mathscr{X}).$$

8.2 Deformation Theory

The notion of square-zero extensions allows one to define a further condition on prestacks.

Consider $x : S \to \mathscr{X}$ a point in a prestack $\mathscr{X}$ and $(T^*S \xrightarrow{\gamma} \mathscr{F}) \in (\mathrm{QCoh}(S)^{\leq -1}_{T^*S}$. This gives a square-zero extension of S:

$$S \hookrightarrow \mathrm{SqZ}(\gamma) := S \underset{S_{\mathscr{F}}}{\sqcup} S$$

as described in Definition 7.3.2.

Definition 8.2.1. A prestack $\mathscr{X}$ is said to be *infinitesimally cohesive* if for all $S \in \mathrm{Sch}^{\mathrm{aff}}$ and $(T^*S \xrightarrow{\gamma} \mathscr{F}) \in (\mathrm{QCoh}(S)^{\leq -1}_{T^*S}$ as above the canonical map:

$$\mathrm{Maps}_{S/}\mathrm{SqZ}(\gamma), \mathscr{X}) \to \mathrm{pt} \underset{\mathrm{Maps}(S_{\mathscr{F}}, \mathscr{X})}{\times} \mathrm{Maps}(S, \mathscr{X})$$

is an isomorphism.

There is a more direct characterization of prestacks that admit a cotangent complex and are infinitesimally cohesive.

Recall that a map $f : S \to T$ of affine schemes is a *nilpotent embedding* if

a) ${}^{c\ell}f : {}^{c\ell}S \to {}^{c\ell}T$ is closed;

b) $\mathscr{I}_{{}^{c\ell}S, {}^{c\ell}T}$ the ideal of definition of ${}^{c\ell}f$ is nilpotent, i.e. $\mathscr{I}^n_{{}^{c\ell}S, {}^{c\ell}T} = 0$ for some $n \geq 1$.

Proposition 8.2.2. *Let $\mathscr{X}$ be a convergent prestacks, then the following are equivalent:*

(1) $\mathscr{X}$ admits a cotangent complex and is infinitesimally cohesive;

(2) $\mathscr{X}$ takes any pushout

$$\begin{array}{ccc} S_1 & \xrightarrow{\jmath} & S_1' \\ \downarrow{\scriptstyle f} & & \downarrow \\ S_2 & \longrightarrow & S_2' \end{array}$$

in Sch^{aff}, where $\jmath$ is a nilpotent embedding, to a pullback diagram of spaces, i.e. the map

$$Maps(S_2', \mathscr{X}) \to Maps(S_1', \mathscr{X}) \underset{Maps(S_1, \mathscr{X})}{\times} Maps(S_2, \mathscr{X})$$

is an isomorphism.

Proposition 8.2.2 is a consequence of the following fact, which says that one can understand a nilpotent embedding of (affine) schemes as a series of square-zero extensions.

Theorem 8.2.3. *Let $f : S \to T$ be a nilpotent embedding of (affine) schemes, then there exists a sequence*

$$S = S_0^0 \hookrightarrow S_0^1 \hookrightarrow \cdots \hookrightarrow S_0^n =: S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow T,$$

such that

(a) each $S_0^i \hookrightarrow S_0^{i+1}$ and $S_j \hookrightarrow S_{j+1}$ has the structure of a square-zero extension;

(b) for every $j \geq 0$, the map $S_j \to T$ induces an isomorphism

$$\tau^{\leq j}(S_j) \xrightarrow{\simeq} \tau^{\leq j}(T).$$

Give proofs for the two results above.

Chapter 9

Deformation theory

9.1 The notion of deformation theory

9.1.1 Affine case

9.1.2 Functor of points case

The combined notions of convergence, admitting a cotangent complex and infinitesimal cohesiveness are so important that one groups them together in the following:

Definition 9.1.1. A prestack $\mathscr{X}$ is said to admit *deformation theory* if it satisfies:

a) $\mathscr{X}$ is convergent, i.e. $\mathscr{X}(S) \xrightarrow{\simeq} \lim_{n\geq 0}\mathscr{X}(\tau^{\leq n}(S))$;

b) $\mathscr{X}$ admits a cotangent complex, i.e. one has an object $T^*\mathscr{X} \in \mathrm{QCoh}(\mathscr{X})^-$;

c) $\mathscr{X}$ is infinitesimally cohesive.

Remark 9.1.2. By Proposition 8.2.2 one has that $\mathscr{X}$ admits deformation theory if and only if it satisfies the following:

a) $\mathscr{X}$ is convergent;

(b)' takes any pushout

$$\begin{array}{ccc} S_1 & \xrightarrow{\jmath} & S_1' \\ \big\downarrow f & & \big\downarrow \\ S_2 & \longrightarrow & S_2' \end{array}$$

in $\mathrm{Sch}^{\mathrm{aff}}$, where $\jmath$ is a nilpotent embedding, to a pullback diagram of spaces.

Remark 9.1.3. Actually, by Theorem 8.2.3, it is enough to check condition (b)' from Remark 9.1.2 for diagrams of affine schemes

$$\begin{array}{ccc} S_1 & \xrightarrow{\jmath} & S_1' \\ \big\downarrow f & & \big\downarrow \\ S_2 & \longrightarrow & S_2' \end{array}$$

where $\jmath$ is a square-zero extension.

Example 9.1.4. (i) Notice that the inclusion $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}$ preserves pushouts by nilpotent embeddings Indeed (...). Thus, any $Z \in \mathrm{Sch}$ admits deformation theory. Why are schemes convergent?

(ii) For $\{\mathscr{X}_i\}_I$ a filtered diagram of prestacks that admit deformation theory, the colimit Why!?

$$\mathscr{X} := \mathrm{colim}_I \mathscr{X}_i$$

as a prestack admits deformation theory. In particular, any ind-scheme admits deformation theory.

Suppose that $\mathscr{X}$ is a prestack that admits deformation theory, we now present some results that use deformation theory to bootstrap a property of the underlying classical prestack ${}^{c\ell}\mathscr{X}$ to $\mathscr{X}$. This section can be seen as a precise justification of the philosophy that "derived geometry" is "classical geometry" + "deformation theory".

Theorem 9.1.5. *Let $\mathscr{X}$ be a prestack that admits deformation theory. Suppose that ${}^{c\ell}\mathscr{X}$ satisfies Zariski (resp. Nisnevich, étale) descent, then so does $\mathscr{X}$.*

Idea of proof. Consider a diagram in ${}^{\leq 1}\mathrm{Sch}^{\mathrm{aff}}$

$$\begin{array}{ccc} S_0' & \longrightarrow & S_0 \\ \downarrow & & \downarrow \\ S_1' & \longrightarrow & S_1 \end{array}$$

where the horizontal morphisms are étale covers and ${}^{c\ell}S_1' \simeq S_0'$ and ${}^{c\ell}S_1 \simeq S_0$. One obtains the following diagram

$$\begin{array}{ccc} \mathscr{X}(S_0) & \xrightarrow{\simeq} & \lim_{\Delta^{\mathrm{op}}} \mathscr{X}((S_0'/S_0)^\bullet) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\overline{\alpha}} \\ \mathscr{X}(S_1) & \longrightarrow & \lim_{\Delta^{\mathrm{op}}} \mathscr{X}((S_1'/S_1)^\bullet) \\ \uparrow & & \uparrow \\ \mathrm{Fib}(\alpha) & \xrightarrow{\beta} & \mathrm{Fib}(\overline{\alpha}) \end{array}$$

By descent of ${}^{c\ell}\mathscr{X}$ the upper horizontal arrow is an isomorphism, hence it is enough to check that β is an isomorphism. Notice that since $\mathscr{X}$ admits deformation theory one has given any point $x \in \mathscr{X}(S_0)$ one has

$$\mathrm{Fib}(\alpha) \underset{\mathscr{X}(S_0)}{\times} \{x\} \simeq \{ \text{ null homotopies of } T_x^*\mathscr{X} \to \mathscr{F} \},$$

for some $\mathscr{F} \in \mathrm{QCoh}(S_0)^{\heartsuit}[2]$ (What is the correct thing here?).

And similarly we have that

$$\mathrm{Fib}(\overline{\alpha}) \underset{\lim_{\Delta^{\mathrm{op}}} \mathscr{X}((S_0'/S_0)^\bullet)}{\times} \{x\} \simeq \mathrm{Tot}(\text{ null-homotopies of } T_x^*\mathscr{X} \to \mathscr{F}^\bullet), \tag{9.1}$$

where $\mathscr{F}^\bullet$ is the co-simplicial object obtained by considering $\mathscr{F}^n := \pi_n^*\mathscr{F}$ where $\pi_n : (S_0'/S_0)^n \to S_0$ is the projection map.

Then one observes that the totalization in (9.1) can be computed at a finite level, so the result follows from the fact that $\mathrm{Hom}(T_x^*\mathscr{X}, -)$ commutes with finite limits.

Make the above argument a bit more precise. ☐

Here is another useful consequence of deformation theory.

Theorem 9.1.6. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a map of prestacks that admit deformation theory, suppose that:*

1) for every $(S \xrightarrow{x} \mathscr{X}) \in Sch^{\mathrm{aff}}_{/\mathscr{X}}$ one has

$$T_y^*\mathscr{Y} \xrightarrow{\simeq} T_x^*\mathscr{X}, \quad y = f \circ x.$$

2) f induces an isomorphism

$$^{\mathrm{c}\ell}f : {}^{\mathrm{c}\ell}\mathscr{X} \xrightarrow{\simeq} {}^{\mathrm{c}\ell}\mathscr{Y}.$$

Then f is an isomorphism.

Proof. By induction it is enough to check that given a square-zero extension $S \hookrightarrow S'$ where $S \in {}^{\mathrm{c}\ell}\mathrm{Sch}^{\mathrm{aff}}$ one has an isomorphism

$$\mathrm{Maps}_{S/}(S', \mathscr{X}) \simeq \mathrm{Maps}_{S/}(S', \mathscr{Y}). \tag{9.2}$$

However, equation (9.2) can be rewritten as:

$$\left\{ \begin{array}{ccc} T^*_x\mathscr{X} & \longrightarrow & \mathscr{F} \\ & \overset{0}{\searrow} & \downarrow \mathrm{id}_{\mathscr{F}} \\ & & \mathscr{F} \end{array} \right\} \simeq \left\{ \begin{array}{ccc} T^*_y\mathscr{Y} & \longrightarrow & \mathscr{F} \\ & \overset{0}{\searrow} & \downarrow \mathrm{id}_{\mathscr{F}} \\ & & \mathscr{F}. \end{array} \right\}$$

Actually turn this into a proof. □

Finally we list the last result that allows one to check the condition of being a prestack locally almost of finite type by checking that only for classical affine schemes and checking a condition on the cotangent spaces.

Theorem 9.1.7. *Let $\mathscr{X}$ be a prestack that admits deformation theory, and suppose that:*

1) ${}^{\mathrm{c}\ell}\mathscr{X} \in {}^{\mathrm{c}\ell}PreStk_{\mathrm{lft}}$, i.e. for any co-filtered diagram $\{T_i\}_I$ of classical affine schemes the map

$$colim_{I^{\mathrm{op}}}\, {}^{\mathrm{c}\ell}\mathscr{X}(T_i) \to {}^{\mathrm{c}\ell}\mathscr{X}(lim_I T_i)$$

is an isomorphism;

2) for every classical point of finite type, i.e. $(T \xrightarrow{x} \mathscr{X}) \in ({}^{\mathrm{c}\ell}Sch^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathscr{X}}$, one has

$$T^*_x\mathscr{X} \in Coh(T).$$

Then $\mathscr{X}$ is locally almost of finite type.

Remark 9.1.8. Check that the second condition is correct. The second condition above can be checked on cohomology, write it down.

Proof. Mention the idea of the proof. □

9.2 Consequences

9.2.1 Connectivity discussion

The following result is the main, and essentially only tool when one is trying to concretely calculate what the cotangent complex is.

First we introduce some notation. Given $\imath : S \hookrightarrow U$ denote a closed embedding of affine schemes, i.e. the underlying morphism of classical affine schemes is a closed embedding. Let

$$\mathscr{I} \to \mathscr{O}_U \to \imath_*\mathscr{O}_S$$

denote the fiber sequence defining the "ideal of definition" $\mathscr{I}$ [1].

[1] The reason for the quotation marks is that, in general, $\mathscr{I}$ is only an object of $\mathrm{QCoh}(T)^{\leq 0}$, so not an ideal in the usual sense.

Construction 9.2.1. Consider the map $\gamma : T^*S \to T^*(S/U)$, defined as the cofiber of the canonical map of cotangent complexes $\imath^* T^* U \to T^*S$. Let $S' := \mathrm{SqZ}(T^*S \to T^*(S/U))$ denote the square-zero of S by $T^*(S/U)[-1]$. Since U is infinitesimally cohesive and the restriction $\gamma|_{T^*_\imath U} : T^*_\imath U \to T^*S \to T^*(S/U)$ vanishes by definition one has a factorization

$$\begin{array}{ccc} S & \xrightarrow{\jmath} & S' \\ \downarrow{\scriptstyle \imath} & \swarrow{\scriptstyle \imath'} & \\ U & & \end{array}$$

We obtain the corresponding morphism of sheaves

$$\mathscr{O}_U \to \imath'_* \mathscr{O}_{S'} \to \imath_* \mathscr{O}_S, \tag{9.3}$$

by passing to fibers everywhere on (9.3) one obtains the fiber sequence

$$\mathrm{Fib}(\mathscr{O}_U \to \imath'_* \mathscr{O}_{S'}) \to \mathrm{Fib}(\mathscr{O}_U \to \imath_* \mathscr{O}_S) \to \mathrm{Fib}(\imath'_* \mathscr{O}_{S'} \to \imath_* \mathscr{O}_S),$$

which in particular gives $\mathscr{I} \to \imath'_* \mathscr{J}$, where $\mathscr{J}$ is the "ideal of definition" of the embedding $S \hookrightarrow S'$. Thus, by adjunction one has

$$(\imath')^* \mathscr{I} \to \mathscr{J}. \tag{9.4}$$

Notice that $\mathscr{J} \simeq \jmath_* T^*(S/U)[-1]$, since one has

$$\mathscr{O}_{S'} \xrightarrow{\simeq} \jmath_* \mathscr{O}_S \underset{\jmath_* \mathscr{O}_S \oplus \jmath_* T^*(S/U)}{\times} \jmath_* \mathscr{O}_S \simeq \jmath_* \mathscr{O}_S \oplus \jmath_* T^*(S/U)[-1]$$

as quasi-coherent sheaves on S'. Thus applying the $(\jmath^*, \jmath_*)$-adjunction to (9.4) and shifting by [1] we obtain the morphism

$$\epsilon_\imath : \imath^* \mathscr{I}[1] \to T^*(S/U). \tag{9.5}$$

Proposition 9.2.2. *Let $n \geq 0$ be a natural number and consider $f : S \to U$ an arbitrary morphism between affine schemes. Assume that $\mathscr{I} \in QCoh(U)^{-n+1}$, then the morphism*

$$\epsilon_f : f^* \mathscr{I}[1] \to T^*(S/U)$$

given in Construction 9.2.1 is $2n$-connectivity, i.e. $Fib(\epsilon_f) \in QCoh(S)$. In particular, one has

$$\tau^{\geq -2n+1}(\epsilon_f) \text{ is an isomorphism } \text{ and } H^{-2n}(\epsilon_f) \text{ is surjective.}$$

Proof. This is [32, Theorem 7.4.3.12]. ☐

Remark 9.2.3. Construction 9.2.1 also makes sense for an arbitrary morphism $fLS \to U$ between affine schemes, so in general one obtains a map $\epsilon_f : f^* \mathscr{I}[1] \to T^*(S/U)$, where we define $\mathscr{I} := \mathrm{Fib}(\mathscr{O}_U \to f_* \mathscr{O}_S)$. However, since for Proposition 9.2.2 on needs a certain connectivity assume on $\mathscr{I}[1]$ to be able to deduce something about the map ϵ_f this situation is where we have the weakest statement. Nevertheless, this can still give us a vector space that surjects onto $H^0(T^*(S/U))$.

Here are a couple of examples of how Proposition 9.2.2 applies.

Example 9.2.4.

(i) Consider $\imath : {}^{\mathrm{c}\ell}S \to S$ the canonical morphism from the underlying classical affine scheme to itself. Since $\mathscr{I} \in \mathrm{QCoh}(S)^{\leq -1}$, one has that

$$\mathscr{I}[1] \to T^*({}^{\mathrm{c}\ell}S/S)$$

is 3-connectivity. In particular, $T^*({}^{\mathrm{c}\ell}S/S) \in \mathrm{QCoh}({}^{\mathrm{c}\ell}S)^{\leq -2}$; which gives that the morphism

$$\imath^* T^* S \to T^{*\mathrm{c}\ell} S$$

is an isomorphism in cohomological degrees -2 and above. This result can be lifted to obtain a comparison between the cotangent complexes of a derived enhancement of any geometric object and that of its classical underlying object (see below Include a reference.).

(ii) Let $f : S \to U$ be a closed embedding of classical affine schemes with ideal of definition $\mathscr{I}$. Then one has

$$\mathscr{I}[1] \xrightarrow{\simeq} T^*(S/U).$$

Chapter 10

Geometric stacks

We will now introduce a notion that cuts part of the category of stacks Stk into more reasonable objects called geometric stacks. Here is a non-exhaustive list of reasons why one would like to do that:

(i) geometric stacks admit representable deformation theory;

(ii) one can define a category of constructible sheaves and a perverse t-structure on any geometric stack locally almost of finite type;

(iii) geometric stacks satisfy the following principle Write a version of Principle 5.3.5 from [26] here.

Idea: we will start with a nice class of stacks, e.g. affine schemes or disjoint unions of such and inductively include objects that can be obtained as quotients of groupoid objects in this category where the structure maps are "nice", e.g. étale or smooth morphisms.

Remark 10.0.1. The definition of geometric stacks that we will take is aligned with Lurie's thesis [26], the reason for that is that it gives a natural form for the representability Theorem (Theorem 11.1.1).

10.1 Definition

The definition is induction, but the case $n = 0$ is slightly different from the rest, in fact we start by requiring an étale cover, so we obtain that 0-geometric objects are akin to algebraic spaces; but after that we allow for smooth covers, hence getting the (higher analogue of) the notion of Artin stacks, instead of the higher analogue of Deligne–Mumford stacks. The reader is instructed to compare this approach with [35] and [16] where different conventions are taken.

Remark 10.1.1. A word about placid ∞-stacks and variations.

Definition 10.1.2. (i) A stack $\mathscr{X}$ is 0-*geometric* if it satisfies

a) ${}^{c\ell}\mathscr{X}$ is 0-truncated;

b) $\mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is affine representable;

c) there exists a 0-*atlas* $f : \mathscr{Z} \to \mathscr{X}$, i.e. f is an étale surjective morphism and $\mathscr{Z} \simeq \sqcup_I Z_i$ is a disjoint union of affine schemes.

(ii) a morphism $f : \mathscr{X} \to \mathscr{Y}$ between prestacks is 0-*geometric* if for every $(S \to \mathscr{Y}) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathscr{Y}}$ the fiber product $\mathscr{X} \underset{\mathscr{Y}}{\times} S$ is 0-geometric;

(iii) a morphism $\mathscr{X} \to S$ from a 0-geometric stack to an affine scheme is *flat* (resp. *smooth*, *étale*) if for some (equivalently for all) atlas(es) $\sqcup_I Z_i \simeq \mathscr{Z} \to \mathscr{X}$, the composite[1]

$$\mathscr{Z} \to \mathscr{X} \to S$$

is flat (resp. smooth, étale);

(iv) a 0-geometric morphism $\mathscr{X} \to \mathscr{Y}$ is *flat* (resp. *smooth*, *étale*) if for every $(S \to \mathscr{Y}) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathscr{Y}}$ the morphism

$$\mathscr{X} \underset{\mathscr{Y}}{\times} S \to S$$

is flat (resp. smooth, étale).

Now let $n \geq 1$ and suppose that the notions of Definition 10.1.2 have been defined for all $k \in \{0, \dots, n-1\}$, then we make the following:

Definition 10.1.3. (i) A stack $\mathscr{X}$ is *n-geometric* if it satisfies

a) $\mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is $(n-1)$-representable;

b) there exists an $(n-1)$-*atlas* $f : \mathscr{Z} \to \mathscr{X}$, i.e. f is a smooth surjective morphism and $\mathscr{Z} \simeq \sqcup_I Z_i$ is $(n-1)$-geometric stack.

(ii) a morphism $f : \mathscr{X} \to \mathscr{Y}$ between prestacks is *n-geometric* if for every $(S \to \mathscr{Y}) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathscr{Y}}$ the fiber product $\mathscr{X} \underset{\mathscr{Y}}{\times} S$ is n-geometric;

(iii) a morphism $\mathscr{X} \to S$ from a n-geometric stack to an affine scheme is *flat* (resp. *smooth*, *étale*) if for some (equivalently for all) atlas(es) $\mathscr{Z} \to \mathscr{X}$, the composite

$$\mathscr{Z} \to \mathscr{X} \to S$$

is flat (resp. smooth, étale);

(iv) an n-geometric morphism $\mathscr{X} \to \mathscr{Y}$ is *flat* (resp. *smooth*, *étale*) if for every $(S \to \mathscr{Y}) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathscr{Y}}$ the morphism

$$\mathscr{X} \underset{\mathscr{Y}}{\times} S \to S$$

is flat (resp. smooth, étale).

There are some compatibilities to be checked about this definition. See [16, Chapter 2, §4.2]. Maybe spell them out as exercises.

Remark 10.1.4. We decide to drop the n from the atlas, specially because one could substitute $\mathscr{Z}$ for a 0-geometric stack in 10.1.3 (i) b) above and obtain the same notion of n-geometric stack.

We let $\mathrm{Stk}^{\mathrm{n-geom.}}$ denote the full subcategory of prestacks generated by n-geometric stacks.

Notice that given $\mathscr{X}$ an n-geometric stack and $f : \mathscr{Z} \to \mathscr{X}$ an atlas, then f is an étale surjection. Thus, one has

Corollary 10.1.5. *For any atlas $\mathscr{Z} \to \mathscr{X}$ where $\mathscr{X}$ is n-geometric one has equivalences:*

$$L(|\mathscr{Z}^\bullet/\mathscr{X}|_{\mathrm{PStk}}) \simeq |\mathscr{Z}^\bullet/\mathscr{X}|_{\mathrm{Stk}} \xrightarrow{\simeq} \mathscr{X}.$$

Corollary 10.1.5 starts to justify the intuition proposed above for the induction definition of n-geometric, i.e. any n-geometric stack $\mathscr{X}$ can be obtained as the quotient, i.e. geometric realization, of a groupoid object $\mathscr{Z}^\bullet/\mathscr{X}$ in $(n-1)$-geometric stacks[2].

[1] We say a morphism $\sqcup_I Z_i \to S$ from a disjoint union of affine schemes to an affine scheme is flat (resp. smooth, étale) if for each $i \in I$ the morphism

$$Z_i \to S$$

is flat (resp. smooth, étale).

[2] Notice that condition (i) a) in Definition 10.1.3 guarantees that each term $\mathscr{Z}^k/\mathscr{X}$ is $(n-1)$-geometric, for $k \geq 1$.

Corollary 10.1.6. *Let $\mathscr{X} \in Stk^{\text{n-geom.}}$, then ${}^{\leq k}\mathscr{X}$ is $(n+k)$-truncated.*

Proof. We proceed by induction on n. For $n = 0$ this holds by definition.

Now, notice that the geometric realization of a simplicial object with value in $(m-1)$-truncated objects in the ∞-category Spc gives an m-truncated space Give a refence for this..

Since ${}^{\leq k}L$ takes m-truncated objects to m-truncated objects Give a reference for this too it is enough to check that

$$ {}^{\leq k}\mathscr{Z}^i/\mathscr{X} \text{ is } (n+k-1)\text{-truncated} $$

for each $i \in \Delta$, but this follows from the inductive hypothesis, since each $\mathscr{Z}^i/\mathscr{X} \in \mathrm{Stk}^{\text{n}-1-\text{geom.}}$. □

The following result finishes the justification of the definiton based in the intuition we gave before.

Proposition 10.1.7. *For $n \geq 1$, let $\mathscr{X}^\bullet$ be a groupoid object in Stk and suppose that*

1) $\mathscr{X}^0$ and $\mathscr{X}^1$ are n-geometric;

2) the morphisms $\mathscr{X}^1 \rightrightarrows \mathscr{X}^0$ are $(n-1)$-geometric and smooth.

Then $\mathscr{X} := |\mathscr{X}^\bullet|_{\mathrm{Stk}}$ is n-geometric.

Proof. We will only give the idea, see [16, Chapter 2, Proposition 4.3.6] for more details.

We first check condition (i) a) of Definition 10.1.3 that is that

$$ |\mathscr{X}|_{\mathrm{Stk}} \to |\mathscr{X}|_{\mathrm{Stk}} \times |\mathscr{X}|_{\mathrm{Stk}} $$

is $(n-1)$-geometric.

Since L takes n-geometric morphisms to n-geometric morphisms Give an argument for this. and preserves finite limits it is enough to check that

$$ |\mathscr{X}|_{\mathrm{PStk}} \to |\mathscr{X}|_{\mathrm{PStk}} \times |\mathscr{X}|_{\mathrm{PStk}} $$

is $(n-1)$-geometric. Given a morphism $S \to |\mathscr{X}|_{\mathrm{PStk}} \times |\mathscr{X}|_{\mathrm{PStk}}$ from an affine scheme, by definition one has a factorization[3]

$$ S \to \mathscr{X}^0 \times \mathscr{X}^0 \to |\mathscr{X}|_{\mathrm{PStk}} \times |\mathscr{X}|_{\mathrm{PStk}} $$

so we have

$$ \begin{aligned} S \underset{|\mathscr{X}|_{\mathrm{PStk}} \times |\mathscr{X}|_{\mathrm{PStk}}}{\times} |\mathscr{X}|_{\mathrm{PStk}} &\simeq S \underset{\mathscr{X}^0 \times \mathscr{X}^0}{\times} (\mathscr{X}^0 \times \mathscr{X}^0) \underset{|\mathscr{X}|_{\mathrm{PStk}} \times |\mathscr{X}|_{\mathrm{PStk}}}{\times} |\mathscr{X}|_{\mathrm{PStk}} \\ \simeq S \underset{\mathscr{X}^0 \times \mathscr{X}^0}{\times} (\mathscr{X}^0 \underset{|\mathscr{X}|_{\mathrm{PStk}}}{\times} \mathscr{X}^0) \\ &\simeq S \underset{(\mathscr{X}^0 \times \mathscr{X}^0)}{\times} \mathscr{X}^1 \end{aligned} $$

and the claim follows from the assumption that $\mathscr{X}^0 \times \mathscr{X}^0 \to \mathscr{X}^1$ is $(n-1)$-geometric.

A similar argument can prove that $\mathscr{X}^0 \to |\mathscr{X}^\bullet|_{\mathrm{Stk}}$ is smooth and surjective.

Finally, consider an atlas $\mathscr{Z}^0 \to \mathscr{X}^0$, i.e. $\mathscr{Z}^0$ is $(n-1)$-geometric and g is smooth and surjective, so the composite $\mathscr{Z}^0 \to \mathscr{X}^0 \to |\mathscr{X}^\bullet|_{\mathrm{Stk}}$ is an atlas of $|\mathscr{X}^\bullet|_{\mathrm{Stk}}$ and we are done. □

In particular, given

$$ Z^1 \rightrightarrows Z^0 \tag{10.1} $$

two smooth schematic morphisms and $Z^\bullet$ a groupoid object extending (10.1) one has that $L(|Z^\bullet|_{\mathrm{PStk}})$ is 1-geometric.

Example 10.1.8. (i) Given a smooth group scheme G one defines

$$ BG := \left| \cdots \quad \rightrightarrows G \times G \rightrightarrows G \rightrightarrows \mathrm{pt} \right| $$

its classifying stack. This is 1-geometric by Proposition 10.1.7.

[3]Since colimits in prestacks are computed pointwise.

(ii) Recall that for any $R \in \mathrm{CAlg}$ we defined the stable ∞-category $\mathrm{Vect}(R)$. It is clear that this assembles to give a prestack:

$$\mathscr{V}\mathrm{ect} : \mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}$$
$$S \mapsto \mathrm{Vect}(S)^{\simeq},$$

where $\mathrm{Vect}(S)^{\simeq}$ denotes the underlying groupoid of $\mathrm{Vect}(S)$. One can check that $\mathscr{V}\mathrm{ect}$ satisfies étale descent. Indeed, this follows easily from the description of $\mathrm{Vect}(S)$ as the smallest stable subcategory of $\mathrm{QCoh}(S)$ containing $\mathscr{O}_S$ and closed under direct sums and retracts.

We claim that $\mathscr{V}\mathrm{ect} \simeq \sqcup_{n\geq 0} \mathscr{V}\mathrm{ect}_n$, where

$$\mathscr{V}\mathrm{ect}_n(S) := \left\{ \mathscr{F} \in \mathrm{QCoh}(S) \mid \mathscr{F}|_{\mathrm{Spec}\,\kappa} \text{ has dimension } n, \text{ for every } \mathrm{Spec}\,\kappa \to S \right\},$$

where κ is a field.

Notice that the morphism

$$\mathrm{pt} \to \mathscr{V}\mathrm{ect}_n$$
$$\mathrm{pt}(S) \mapsto \mathscr{O}_S^{\oplus n}$$

that maps to the trivial vector bundle of rank n on S is an étale surjection. Indeed, when restricted to the underlying classical prestacks it follows from the definition of vector bundles on an affine scheme that they are locally trivial for the Zariski topology, so also in the étale topology. To check this in the derived setting one uses the classical result plus the existence of homotopy inverses for projective modules (see [35, Proposition 2.9.2.3] for details). What do I mean by this?. Thus,

$$|(\mathrm{pt}/\mathscr{V}\mathrm{ect}_n)^{\bullet}| \simeq \mathscr{V}\mathrm{ect}_n$$

for any $n \geq 0$. Now we claim that:

$$\mathrm{pt} \underset{\mathscr{V}\mathrm{ect}_n}{\times} \mathrm{pt} \simeq GL_n.$$

Given the claim one has that $\mathscr{V}\mathrm{ect}_n$ is 1-geometric, since GL_n is affine.

Remark 10.1.9. Toën has two important results concerning the class of n-geometric stacks Give a citation.:

(a) any n-geometric stack satisfy flat descent;

(b) if one required only flat descent (instead of étale descent) and a flat atlas (instead of condition (i) b)) in Definition 10.1.3 one would obtain the same class of objects. Maybe make this a bit more precise, what happens in the case $n = 0$?

Note this is similar to what happens in the usual definition of algebraic spaces/stacks. Give a reference!

10.2 Properties

The following collects some consequences of being a geometric stack.

Proposition 10.2.1. *Consider $\mathscr{X}$ an n-geometric stack, then*

(a) $\mathscr{X}$ admits deformation theory and $T^*\mathscr{X} \in QCoh(\mathscr{X})^{\leq n}$;

(b) let $\mathscr{X} \to \mathscr{Y}_0$ be a smooth morphism where $\mathscr{Y}_0$ is 0-geometric, then for any $x : S \to \mathscr{X}$ one has $T^*_x(\mathscr{X}/\mathscr{Y}_0) \in QCoh(S)^{\geq 0, \leq (n-1)}$.

Idea of proof. Consider $f : \mathscr{Z} \to \mathscr{X}$ and assume that

- $\mathscr{X}$ satisfies étale descent;

- f is an étale surjection;
- $\mathscr{Z}$ admits deformation theory;
- f admits deformation theory;
- f is formally smooth Definition of this!.

We will prove that $\mathscr{X}$ admits deformation theory. Consider $S_2' := S_1' \underset{S_1}{\sqcup} S_2$ where $S_1 \hookrightarrow S_1'$ has the structure of a square-zero extension. We need to show that for any $x_2 : S_2 \to \mathscr{X}$ the morphism

$$\mathscr{X}(S_2') \underset{\mathscr{X}(S_2)}{\times} \mathrm{pt} \to \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathrm{pt}$$

is an equivalence.

By étale descent the statement is étale local on S_2, so we can assume that $x_2 : S_2 \to \mathscr{X}$ lifts to a morphism $z_2 : S_2 \to \mathscr{Z}$. Thus we have

$$\begin{array}{ccc} \left|(\mathscr{Z}/\mathscr{X})^{\bullet}(S_2') \underset{(\mathscr{Z}/\mathscr{X})^{\bullet}(S_2)}{\times} \mathrm{pt}\right| & \longrightarrow & \mathscr{X}(S_2') \underset{\mathscr{X}(S_2)}{\times} \mathrm{pt} \\ \downarrow & & \downarrow \\ \left|(\mathscr{Z}/\mathscr{X})^{\bullet}(S_1') \underset{(\mathscr{Z}/\mathscr{X})^{\bullet}(S_1)}{\times} \mathrm{pt}\right| & \longrightarrow & \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathrm{pt}. \end{array}$$

We notice that the horizontal arrows are monomorphisms, since one has a lift as in the dotted arrow:

$$\begin{array}{ccc} S_2 & \longrightarrow & \mathscr{Z} \\ \downarrow & \nearrow & \downarrow f \\ S_2' & \longrightarrow & \mathscr{X} \end{array}$$

by formal smoothness of f. Why!? Is that so?

So it is enough to prove that these maps induce a surjection on connected component, i.e. that they are essentially surjective as morphism of spaces, since being a monomorphism of spaces implies that the map is fully faithful. We claim that this will follow from f being formally smooth, i.e. for all $(S \xrightarrow{x} \mathscr{Z}) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathscr{Z}}$ one has

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(T_x^*(\mathscr{Z}/\mathscr{X}), \mathscr{F}) \in \mathrm{Vect}^{\leq 0}, \quad \forall \mathscr{F} \in \mathrm{QCoh}(S)^{\heartsuit}. \tag{10.2}$$

Given a diagram

$$\begin{array}{ccc} S & \xrightarrow{z} & \mathscr{Z} \\ \downarrow g & \nearrow & \downarrow f \\ S' & \longrightarrow & \mathscr{X} \end{array}$$

where g has the structure of a square-zero extension we claim that the dotted arrow exists. Indeed, let $\gamma : T^*S \to \mathscr{F}$ with $\mathscr{F} \in \mathrm{QCoh}(S)^{\leq -1}$ be the morphism witnessing g as a square-zero extension, the space of such γ is isomorphic to the space of null-homotopies of $T_z^*(\mathscr{Z}/\mathscr{X}) \to \mathscr{F}$, which is non-empty because of (10.2).

By inducing on n one obtains that any n-geometric stack admits deformation theory. Let's now calculate the connectivity of their cotangent complex.

For $n = 0$, for any $z : S \to \mathscr{Z}$ one has a cofiber/fiber sequence

$$T_{f\circ z}^*(\mathscr{X}) \to T_z^*\mathscr{Z} \to T_z^*(\mathscr{Z}/\mathscr{X}). \tag{10.3}$$

Since $f : \mathscr{Z} \to \mathscr{X}$ is étale, by Reference to this statement one has that $T_z^*(\mathscr{Z}/\mathscr{X}) \simeq 0$. Since $\mathscr{Z}$ is a disjoint union of affine schemes onw has $T_z^*\mathscr{Z}$ is connective, which implies that $T_{f\circ z}^*(\mathscr{X})$ is connective. Since the morphism f is surjective one obtains that $T_x^*\mathscr{X}$ is connective for any $x : S \to \mathscr{X}$.

For $n = 1$, using the again the sequence (10.3), this time one has $T^*_x(\mathscr{Z}/\mathscr{X}) \in \mathrm{QCoh}(S)^{\heartsuit}$, since $\mathscr{Z} \to \mathscr{X}$ is smooth hence also formally smooth. This gives that $T^*_x\mathscr{X} \in \mathrm{QCoh}(S)^{\leq 1}$ for every $x : S \to \mathscr{X}$. Give the reference/argument for the connection between the condition of formal smoothness on the cotangent complex and the connectivity. Also give a reference to smooth implies formally smooth.

By induction one has the statement for $n \geq 0$. □

10.2.1 Geometric stacks

Let $\mathscr{X}$ be a locally geometric stack, i.e. $\mathscr{X} := \mathrm{colim}_I \mathscr{X}_i$ where each $\mathscr{X}$ is n_i-geometric for some $n_i \in \mathbb{N}$.

10.3 Example: Perfect complexes

In this section we will prove that the stack of perfect complex is locally geometric. We will follow the arguments in [55].

First we need to discuss some homological algebra for perfect R-modules.

Definition 10.3.1. Given two integers $a \leq b$ and $M \in \mathrm{Mod}_R$ we say that M has *Tor-amplitude in* $[a, b]$ if for any $N \in \mathrm{Mod}_R^{\heartsuit}$ one has

$$M \otimes_R N \in \mathrm{Mod}_R^{\geq a, \leq b}.$$

We will denote by $(\mathrm{Mod}_R)_{[a,b]}$ the full subcategory of R-modules with Tor-amplitude in $[a, b]$ and $\mathrm{Perf}(R)_{[a,b]} := \mathrm{Perf}(R) \cap (\mathrm{Mod}_R)_{[a,b]}$ the full subcategory of perfect R-modules with Tor-amplitude in $[a, b]$.

Here is a list of facts about $\mathrm{Perf}(R)_{[a,b]}$ that we will need in the discussion of the stack of perfect complexes.

Proposition 10.3.2. *Let $a \leq b$ and $c \leq d$ be two pairs of integers.*

(a) Given $P \in \mathrm{Perf}(R)_{[a,b]}$ and $Q \in \mathrm{Perf}(R)_{[c,d]}$ one has

$$P \otimes_R Q \in \mathrm{Perf}(R)_{[a+c,b+d]};$$

(b) for $f : P \to Q$ a morphism in $\mathrm{Perf}(R)_{[a,b]}$ one has $\mathrm{Fib}(f) \in \mathrm{Perf}(R)_{[a,b+1]}$;

(c) $P \in \mathrm{Perf}(R)_{[a,b]}$ if and only if $P \otimes_R H^0R \in \mathrm{Perf}(H^0(R))_{[a,b]}$;

(d) for any ring map $R \to R'$, the functor $(-) \otimes_R R'$ restricts to a functors

$$(-) \otimes_R R' : \mathrm{Perf}(R)_{[a,b]} \to \mathrm{Perf}(R')_{[a,b]};$$

(e) $\mathrm{Perf}(R) = \bigcup_{a \leq b} \mathrm{Perf}(R)_{[a,b]}$;

(f) $\mathrm{Perf}(R)_{[a,a]} \simeq \mathrm{Vect}(R)[-a]$;

(g) for any $P \in \mathrm{Perf}(R)_{[a,b]}$, there exists $E \in \mathrm{Vect}(R)$ and a map

$$E[-b] \xrightarrow{f} P \to \mathrm{Cofib}\, f$$

such that $\mathrm{Cofib}\, f \in \mathrm{Perf}(R)_{[a,b-1]}$.

Idea of the proof. (i-iii) are clear from the definition.

(iv) follows from the fact that $\mathrm{oblv}_{R' \to R} : \mathrm{Mod}_{R'} \to \mathrm{Mod}_R$ is t-exact.

(v) follows from (iii) and the classical result.

(vi) given $P \in \mathrm{Perf}(R)_{[a,b]}$ one has that $P[a]$ is flat and almost perfect, so [32, Proposition 7.2.4.20] implies that $P[a] \in \mathrm{Vect}(R)$.

(vii) consider $E_0 \in \mathrm{Vect}(H^0R)$ such that one has a cofiber/fiber sequence of $H^0(R)$-modules:

$$E_0[b] \xrightarrow{\overline{f}} P \otimes_R H^0R \to \mathrm{Cofib}\, \overline{f}$$

with $\mathrm{Cofib}\, \overline{f} \in \mathrm{Perf}(H^0R)_{[a,b]}$. Then one can find $E \in \mathrm{Vect}(R)$ such that $E_0 \simeq E \otimes_R H^0R$. Indeed, this is clear for free $H^0(R)$-modules, the general case follows from using that retracts in Mod_R are given by retracts in Mod_R. Thus, one has a lift $f : E[-b] \to P \otimes_R H^0(R)$ and one can directly check that $\mathrm{Cofib}\, f \in \mathrm{Perf}(R)_{[a,b-1]}$ as required. □

Consider the prestack

$$\mathscr{M} : \mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}$$
$$S \mapsto \mathrm{Perf}(S)^{\simeq}.$$

We start by noticing:

Proposition 10.3.3. *$\mathscr{M}$ is a stack satisfies flat descent.*

Proof. Since the prestack $R \mapsto (\mathrm{Mod}_R)^{\simeq}$ satisfies flat descent it is enough to check that the condition of being perfect is local for the flat topology.

Given $R \to R'$ a flat morphism and $M \in \mathrm{Mod}_R$ such that $M' := M \otimes_R R'$ is a perfect R'-module, we claim that M is a perfect R-module. Indeed, for an R'-module being perfect is equivalent to dualizable Give a reference., so let $(M')^{\vee}$ denote the dual of M', i.e. one has an equivalence:

$$(M')^{\vee} \otimes_{R'} (-) \simeq \mathrm{Hom}_{\mathrm{Mod}_{R'}}(M', -).$$

By descent for Mod_R we let $M^{\vee}$ be the R-module such that $M^{\vee} \otimes_{R'} R \simeq (M')^{\vee}$. Then $M^{\vee}$ exhibits a dual of M, hence M is perfect. □

Lemma 10.3.4. *Let $b \geq 0$ and consider $\mathscr{F} \in \mathit{Perf}(S)_{[a,b]}$, then the stack:*

$$\mathit{Sect}(\mathbb{V}(\mathscr{F})) : \mathit{Sch}^{\mathrm{aff}}_{/S} \to \mathit{Spc}$$
$$(T \xrightarrow{f} S) \mapsto \mathit{Hom}_{\mathit{QCoh}(S)}(\mathscr{F}, f_*\mathscr{O}_T)$$

is b-geometric.

Proof. First we consider the case $b = 0$, we claim that the relative spectrum of the object $\mathrm{Sym}(\mathscr{F})$ of $\mathrm{CAlg}(\mathrm{QCoh}(S))_{\mathscr{O}_S/}$ represents $\mathrm{Sect}(\mathbb{V}(\mathscr{F}))$. Indeed, one has

$$\begin{aligned}\mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathscr{F}, f_*\mathscr{O}_T) &\simeq \mathrm{Hom}_{\mathrm{QCoh}(T)}(f^*\mathscr{F}, \mathscr{O}_S)\\ &\simeq \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{QCoh}(T))_{\mathscr{O}_T/}}(\mathrm{Sym}(f^*\mathscr{F}), \mathscr{O}_T)\\ &\simeq \mathrm{Hom}_{\mathrm{Sch}^{\mathrm{aff}}}(T, \mathrm{Spec}(\mathrm{Sym}(f^*\mathscr{F}))).\end{aligned}$$

Get the equation right above! This is a matter of understanding the relative construction.

Since $\mathrm{Sect}(\mathbb{V}(\mathscr{F})) \simeq \mathrm{Spec}(\mathrm{Sym}(\mathscr{F}))$, one obtains that $\mathrm{Sect}(\mathbb{V}(\mathscr{F}))$ is affine, hence 0-geometric.

Inductive step: suppose that for all $a \leq b-1$ and $\mathscr{F} \in \mathrm{Perf}(S)_{[a,b-1]}$ we proved that $\mathrm{Sect}(\mathbb{V}(\mathscr{F}))$ is $(b-1)$-geometric. Consider $\mathscr{G} \in \mathrm{Perf}(S)_{[a,b]}$ and let $\mathscr{E} \in \mathrm{Vect}(S)$ and $\mathscr{F}\mathrm{Perf}(S)_{[a,b]}$ be such that one has an exact sequence:

$$\mathscr{E}[-b] \to \mathscr{G} \to \mathscr{F}.$$

Then for $T \xrightarrow{f} S$ one has

$$\begin{aligned}\mathrm{Sect}(\mathbb{V}(\mathscr{G}))(T) &\simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathscr{G}, f_*\mathscr{O}_T)\\ &\simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathrm{Cofib}(\mathscr{F}[-1] \to \mathscr{E}[-b]), f_*\mathscr{O}_T)\\ &\simeq \mathrm{Fib}\left(\mathrm{Hom}(\mathscr{E}[-b+1], f_*\mathscr{O}_T)[1] \to \mathrm{Hom}(\mathscr{F}, f_*\mathscr{O}_T)[1]\right),\end{aligned}$$

so

$$\Omega(\mathrm{Sect}(\mathbb{V}(\mathscr{G}))) \simeq \mathrm{Fib}\left(\mathrm{Sect}(\mathbb{V}(\mathscr{E}[-b+1])) \to \mathrm{Sect}(\mathbb{V}(\mathscr{F}))\right).$$

Explain a bit better what $\Omega(\mathrm{Sect}(\mathbb{V}(\mathscr{G})))$ is.

Now we notice that Exercise! $\mathrm{Sect}(\mathbb{V}(\mathscr{E}[-b+1])) \simeq B^{b-1}\mathbb{G}_{\mathrm{a}}^r$ where $r = \mathrm{rank}\mathscr{E}$ and we iterate the delooping construction $(b-1)$ times. In particular, one has that $\mathrm{Sect}(\mathbb{V}(\mathscr{E}[-b+1]))$ is $(b-1)$-geometric and by the inductive hypothesis so is $\mathrm{Sect}(\mathbb{V}(\mathscr{F}))$. This implies that $\Omega(\mathrm{Sect}(\mathbb{V}(\mathscr{G})))$ is $(b-1)$-geometric.

Since

$$\mathrm{Sect}(\mathbb{V}(\mathscr{G})) \simeq |(\Omega(\mathrm{Sect}(\mathbb{V}(\mathscr{G})))/\mathrm{pt})^{\bullet}|$$

the result follows from Proposition 10.1.7. □

Theorem 10.3.5. *For integers $a \leq b$ consider the prestack $\mathscr{M}_{[a,b]}$ whose S-points are:*

$$\mathscr{M}_{[a,b]}(S) := (Perf(S)_{[a,b]})^{\simeq}.$$

The prestack $\mathscr{M}_{[a,b]}$ is $(b-a+1)$-geometric.

Proof. Let $n = b-a+1$.

First we check that $\mathscr{M}_{[a,b]}$ has $(n-1)$-geometric diagonal. It is enough to prove that for any pair of points $x : S \to \mathscr{M}_{[a,b]}$ and $y : T \to \mathscr{M}_{[a,b]}$ the pullback

$$S \underset{\mathscr{M}_{[a,b]}}{\times} T \text{ is } (n-1)\text{-geometric.}$$

Notice that

$$S \underset{\mathscr{M}_{[a,b]}}{\times} T \simeq \left\{ \alpha \in \mathrm{Hom}_{\mathrm{QCoh}(S\times T)}(p_1^*\mathscr{F}, p_2^*\mathscr{G}) \mid \alpha \text{ is an automorphism} \right\},$$

where $\mathscr{F} \in \mathrm{Perf}(S)_{[a,b]}$ corresponds to x and $\mathscr{G} \in \mathrm{Perf}(T)_{[a,b]}$ corresponds to y. Given an element $\alpha \in \mathrm{Hom}_{\mathrm{QCoh}(S\times T)}(p_1^*\mathscr{F}, p_2^*\mathscr{G}) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S\times T)}(p_1^*\mathscr{F} \otimes (p_2^*\mathscr{G})^\vee, \mathscr{O}_{S\times T})$. Since

$$\mathrm{Hom}_{\mathrm{QCoh}(S\times T)}(p_1^*\mathscr{F} \otimes (p_2^*\mathscr{G})^\vee, \mathscr{O}_{S\times T}) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(p_1^*\mathscr{F}, \mathscr{O}_S) \otimes \mathrm{Hom}_{\mathrm{QCoh}(T)}(p_2^*(\mathscr{G})^\vee, \mathscr{O}_T),$$

one obtains a map

$$\jmath : S \underset{\mathscr{M}_{[a,b]}}{\times} T \to \mathrm{Sect}(\mathbb{V}(\mathscr{F} \boxtimes \mathscr{G}^\vee)).$$

By Proposition 10.3.2 $\mathscr{F} \boxtimes \mathscr{G}^\vee \in \mathrm{Perf}(S \times T)_{[a-b,b-a]}$, so Lemma 10.3.4 gives that $\mathrm{Sect}(\mathbb{V}(\mathscr{F} \boxtimes \mathscr{G}^\vee))$ is $(n-1)$-geometric. The result will follow if we prove that $\jmath$ is $(n-1)$-geometric. In fact, we can do better and prove that $\jmath$ is affine. Indeed, by considering the following pullback diagrams

$$\begin{array}{ccc} S \underset{\mathscr{M}_{[a,b]}}{\times} T & \xrightarrow{\ \jmath\ } & \mathrm{Sect}(\mathbb{V}(\mathscr{F} \boxtimes \mathscr{G}^\vee)) \\ \downarrow & & \downarrow \\ \tau_{\leq 0}(S \underset{\mathscr{M}_{[a,b]}}{\times} T) & \xrightarrow{\tau_{\leq 0}(\jmath)} & \tau_{\leq 0}(\mathrm{Sect}(\mathbb{V}(\mathscr{F} \boxtimes \mathscr{G}^\vee))) \end{array}$$

where $\tau_{\leq 0}\mathscr{X}$ denotes the 0-truncation of the stack $\mathscr{X}$. We claim that $\tau_{\leq 0}(\jmath)$ is affine representable. Why is the bottom morphism affine representable and the diagram is a pullback?

Now we check that $\mathscr{M}_{[a,b]}$ admits an atlas, i.e. we need to construct an $(n-1)$-geometric stack $\mathscr{Z}$ and a smooth and surjective morphism $\mathscr{Z} \to \mathscr{M}_{[a,b]}$.

Naturally, we proceed by induction on $n = b-a+1$. The case $n=1$ corresponds to $\mathscr{M}_{[a,a]} \simeq \mathscr{V}\mathrm{ect}$ and was discussed in Example 10.1.8 (ii).

Consider the stack $\mathscr{U}$ defined as the following pullback diagram:

$$\begin{array}{ccc} \mathscr{U} & \longrightarrow & \mathrm{Fun}([1], \mathrm{Perf})^{\simeq} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0 \times \mathrm{ev}_1} \\ \mathscr{M}_{[a,b-1]} \times \mathscr{V}\mathrm{ect} & \xrightarrow{\ f\ } & \mathscr{M} \times \mathscr{M} \end{array}$$

where the bottom morphism is $f(M,N) = (M, N[-b+1])$. Concretely, one has

$$\mathscr{U}(S) = \left\{ M \in \mathrm{Perf}(S)_{[a,b-1]},\ N \in \mathrm{Vect}(S),\ \varphi : M \to N[-b+1] \right\}.$$

Notice that one has a morphism $p : \mathscr{U} \to \mathscr{M}_{[a,b]}$ given by

$$p((M,N,\varphi)) = \mathrm{Fib}\,\varphi$$

since by Proposition 10.3.2 $\operatorname{Fib}\varphi$ has Tor-amplitude in $[a,b]$. We need to prove: (i) $\mathscr{U}$ is $(n-1)$-geometric and (ii) p is smooth and surjective.

For (i) we consider $\pi : \mathscr{U} \to \mathscr{M}_{[a,b-1]} \times \mathscr{V}\mathrm{ect}$ in the diagram defining $\mathscr{U}$. By induction we know that $\mathscr{M}_{[a,b-1]} \times \mathscr{V}\mathrm{ect}$ is $(n-1)$-geometric, so it is enough to check that π is $(n-1)$-geometric. Given an affine point $x : S \to \mathscr{M}_{[a,b-1]} \times \mathscr{V}\mathrm{ect}$ corresponding to $(M,N) \in \mathscr{M}_{[a,b-1]}(S) \times \mathrm{Vect}(S)$ we have

$$\pi^{-1}(S) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(M, N[-b+1]) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(M \otimes N^{\vee}[b-1], \mathscr{O}_S),$$

since $M \otimes N^{\vee}[b-1] \in \mathscr{M}_{[a-b+1,0]}$, one has that $\pi^{-1}(S)$ is affine, so we are done.

For (ii) is is clear that p is surjective, by Proposition 10.3.2. We recall that p is smooth if for every $x : S \to \mathscr{M}_{[a,b]}$ the pullback $S \underset{\mathscr{M}_{[a,b]}}{\times} \mathscr{U} \to S$ is smooth as an $(n-1)$-geometric stack.

Consider the pullback diagram

$$\begin{array}{ccccc}
S \underset{S\times\mathscr{V}\mathrm{ect}}{\times} (S \underset{\mathscr{M}_{[a,b]}}{\times} \mathscr{U}) & & \longrightarrow & & S \underset{\mathscr{M}_{[a,b]}}{\times} \mathscr{U} \\
 & & & & \downarrow \\
\Big\downarrow q & & & & S \times \mathscr{U} \\
 & & & & \downarrow \mathrm{id}_{S\times\pi} \\
 & & & & S \times \mathscr{M}_{[a,b-1]} \times \mathscr{V}\mathrm{ect} \\
 & & & & \downarrow \\
\coprod_{r\geq 0} S & \longrightarrow & \coprod_{r\geq 0} S \times \mathscr{V}\mathrm{ect}_r & \xrightarrow{\simeq} & S \times \mathscr{V}\mathrm{ect},
\end{array}$$

and let V_r denote the fiber of q over the rth copy of S and $q_r : V_r \to S$ the induce morphism. Since the bottom map in the above diagram is a smooth cover, it is enough to check that each q_r is smooth for each $r \geq 0$.

Notice that for $(T \xrightarrow{f} S)$ one has

$$V_r \underset{S}{\times} T \simeq \mathrm{Hom}_{\mathrm{QCoh}(T)})(\mathscr{O}_T^r[-b+1], f^*P[1]),$$

where $P \in \mathrm{Perf}(S)_{[a,b]}$ corresponds to the point x fixed above. Since one has

$$\mathrm{Hom}_{\mathrm{QCoh}(T)})(\mathscr{O}_T^r[-b+1], f^*P[1]) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathscr{O}_S^r \otimes P^{\vee}[-b], \mathscr{O}_T)$$

for any sheaf $\mathscr{F} \in \mathrm{QCoh}(T)^{\leq 0}$ one has the equivalences:

$$\mathrm{Maps}_{T/}(T_{\mathscr{F}}, V_r) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathscr{O}_S^r \otimes P^{\vee}[-b], f_*\mathscr{F}) \simeq \mathrm{Hom}_{\mathrm{QCoh}(T)}(\mathscr{O}_T^r \otimes f^*P^{\vee}[-b], \mathscr{F}).$$

Thus, $T^*(V_r/S) \simeq \mathscr{O}_T^r \otimes f^*P^{\vee}[-b] \in \mathrm{Perf}(T)_[0, b-a]$.

This implies that q_r is formally smooth, since it is also locally almost of finite type it is also smooth Reference for this, again!.

This finishes the proof. □

Corollary 10.3.6. *$\mathscr{M}$ is a locally geometric stack, i.e. $\mathscr{M} \simeq \bigcup_{n\geq 0} \mathscr{M}^{(n)}$ where each $\mathscr{M}^{(n)}$ is n-geometric and locally almost of finite type.*

Proof. □

Chapter 11

Representability Theorem

11.1 Lurie's Theorem

We are finally ready to prove the following result due to J. Lurie that characterizes n-geometric stacks.

Theorem 11.1.1. *Given a prestack $\mathscr{X}$, $\mathscr{X}$ is an n-geometric stack locally almost of finite type if and only if the following conditions are satisfied:*

1) $\mathscr{X}$ is locally almost of finite type as a prestack;

2) $\mathscr{X}$ satisfies étale descent;

3) for any discrete complete local Noetherian k-algebra R_0 with maximal ideal $\mathfrak{m}_0 \subset R_0$ the natural map

$$^{c\ell}\mathscr{X}(Spec(R_0)) \to lim_{n\geq 1}{}^{c\ell}\mathscr{X}(Spec(R_0/\mathfrak{m}_0^n))$$

is an isomorphism;

4) $\mathscr{X}$ admits $(-n)$-connective representable deformation theory, i.e. $T^*\mathscr{X}_x \in QCoh(S)^{\leq n}$ for every affine point $x : S \to \mathscr{X}$;

5) $^{c\ell}\mathscr{X}$ is n-truncated.

Before we prove this theorem here are some comments on it.

Remark 11.1.2. The theorem as stated is close to [26, Theorem 7.1.6], similar versions were proved latter in the DAG series of papers, and there is still no version in the spectral setting, but that is expected to be in [35, Chapter 27].

Remark 11.1.3. There are two classical versions of this theorem due to Artin []. The first characterizes which functors $\mathscr{X}_0 : {}^{c\ell}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}^{\leq 0}$ are classical algebraic spaces a modern reference for this is [Stacks, Tag 07XZ and Tag 07Y0]. The second considers which functors $\mathscr{X}_0 : {}^{c\ell}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}^{\leq 1}$ are algebraic stacks.

Since Artin's work there has been some significant improvement on the list of axioms and clarification of the role of certain conditions. We recommend [1, Lectures 1 and 2] for a great review and the source article [20] for a very well explained proof and comparison with other similar statements in the literature.

Remark 11.1.4. To what extent is it true that $\mathscr{X}_0$ admits a derived enhacement $\mathscr{X}$ that satisfies condition (iv) is equivalent to $\mathscr{X}_0$ satisfying conditions a), b) and c) from the above Remark?

Remark 11.1.5. There is also a notion of a complete local Noetherian (derived) ring $R \in \mathrm{CAlg}$, i.e.

- R is a Noetherian (derived) ring;
- there exists an unique $\mathfrak{m}_0 \subset H^0(R)$;
- $H^0(R)$ is $\mathfrak{m}_0$-adically complete, i.e. the natural map $H^0(R) \to \lim_{n\geq 1} \frac{H^0(R)}{\mathfrak{m}_0^n}$ is an isomorphism.

One can then prove (see [35, Proposition 7.1.7]) that condition (iii) is equivalent to:

$(iii)^{\mathrm{der}}$ for every complete local Noetherian ring $R \in \mathrm{CAlg}$ one has:

$$\mathscr{X}(\mathrm{Spec}(R)) \to \lim_{n\geq 1}\mathscr{X}(\mathrm{Spec}(R_n))$$

is an isomorphism, where $R_n := R \otimes_{\mathbb{Z}[y_1,\dots,y_m]} \mathbb{Z}$ where $\{x_1,\dots,x_m\}$ is a set of generators of $\mathfrak{m}_0$ and y_i are lifts of $x_i^{2^n}$ to R.

Remark 11.1.6. Since $\mathscr{X}$ admits deformation theory by applying [17] we can relax condition (i) to:

$(i)^{\mathrm{cl}}$ ${}^{\mathrm{c}\ell}\mathscr{X}$ is locally of finite type;

if we also impose

$(iv)^{\mathrm{ft}}$ for every $x : S \to \mathscr{X}$ with $S \in {}^{\mathrm{c}\ell}\mathrm{Sch}^{\mathrm{aff}}{}_{\mathrm{ft}}$ one has $T^*_x\mathscr{X} \in \mathrm{Coh}(S)^-$.

11.2 Comparison with classical results

In this section we compare Theorem 11.1.1 with results that characterize when a groupoid-valued functor on classical affine schemes is representable by a classical (Artin) algebraic stack.

The following is the main result of [21].

Theorem 11.2.1. *Consider k an excellent classical affine scheme $\mathsf{X} : {}^{\mathrm{c}\ell}Sch^{\mathrm{aff}}{}_k \to Spc^{\leq 1}$ a 1-truncated classical prestack then X is a classical (Artin) algebraic stack locally of finite presentation if and only if the following conditions are satisfied*

1) X is locally of finite type as a classical prestack;

2) X is an étale stack;

3) for any diagram of classical affine schemes $SpecB \leftarrow SpecA \xrightarrow{\imath} SpecA'$ where $\imath$ is a nilpotent closed embedding, the natural map

$$\mathsf{X}(Spec(B\otimes_A A')) \to \mathsf{X}(Spec(B)) \underset{\mathsf{X}(Spec(A))}{\times} \mathsf{X}(Spec(A'))$$

is an isomorphism;

4) for any discrete complete local Noetherian k-algebra R_0 with maximal ideal $\mathfrak{m}_0 \subset R_0$ the natural map

$$\mathsf{X}(Spec(R_0)) \to lim_{n\geq 1}\mathsf{X}(Spec(R_0/\mathfrak{m}_0^n))$$

is an isomorphism;

5)

11.3 Homogeneity

In this section we investigate the existence of the (pro-)cotangent complex from a weaker condition than that considered in §**??**.

We start by recalling the following started fact about morphisms of ∞-groupoids.

Lemma 11.3.1. *Let $F : S \to T$ be a morphism in the category Spc then*

(i) F is essentially surjective (as a morphism of ∞-categories) if and only if

$$\pi_0(F) : \pi_0(S) \to \pi_0(T) \quad \textit{is surjective.}$$

(ii) F is fully faithful (as a morphism of ∞-categories) if and only if

$$\pi_0(F) : \pi_0(S) \to \pi_0(T) \quad \textit{is surjective, and}$$
$$\forall s \in \pi_0(S) \quad \pi_(S, s) \to \pi_i(T, F(s)) \quad \textit{is bijective, for } i \geq 1.$$

Proof. By definition F is essentially surjective if for every $t \in T$ there exists $s \in S$ and an isomorphism $F(s) \simeq t$. If F is essentially surjective for any $\bar{t} \in \pi_0(T)$, let $t \in T$ be a representative, then for $s \in S$ with $F(s) \simeq t$ one clearly has $\pi_0(F)(\bar{s}) = \overline{F(s)} = \bar{t}$, so $\pi_0(F)$ is surjective.

Now suppose that $\pi_0(F)$ is surjective. Let $t \in T$ one knows that there exists $\bar{s} \in \pi_0(S)$ such that $\pi_0(F)(\bar{s}) = \bar{t}$ in $\pi_0(T)$. Let $s \in S$ be a lift of $\bar{s}$ to S, by definition of the induced morphism $\pi_0(F)$ one has $\overline{F(s)} = \pi_0(F)(\bar{s}) = \bar{t}$. Since two elements of T produce the same class if and only if there exists a 1-morphism $\alpha : F(s) \to t$ between them, since T is a groupoid α is automatically an isomorphism.

Recall that F is fully faithful if for every pair of objects $s, s' \in S$ one has an equivalence of spaces

$$\mathrm{Hom}_S(s, s') \xrightarrow{\simeq} \mathrm{Hom}_T(F(s), F(s')). \tag{11.1}$$

First we notice that if $\pi_0(F)$ is not injective, then one can find $s, s' \in S$ such that $s \not\simeq s'$ and $F(s) \simeq F(s')$. Since any morphism in S is invertible if $s \not\simeq s'$ then $\mathrm{Hom}_S(s, s') = \emptyset$, so equation (11.1) gives

$$\emptyset = \mathrm{Hom}_S(s, s') \xrightarrow{\simeq} \mathrm{Hom}_T(F(s), F(s')) = \mathrm{pt},$$

which is a contraction.

Because $\pi_0(F)$ needs to be injective, we can assume that $s, s' \in S$ such that $s = s'$, then one has

$$\mathrm{Hom}_S(s, s') \simeq \mathrm{Hom}_S(s, s) \simeq \mathrm{pt} \underset{S}{\times} \mathrm{pt}$$

where $s : \mathrm{pt} \to S$ are the maps we are taking the pullback with respect to, and similarly $F(s) \simeq F(s')$. Hence condition (11.1) is equivalent to

$$\Omega_s(S) := \mathrm{pt} \underset{S}{\times} \mathrm{pt} \xrightarrow{\simeq} \mathrm{pt} \underset{T}{\times} \mathrm{pt} =: \Omega_{F(s)}T$$

which happens if and only if[1]

$$\pi_{k+1}(S, s) = \pi_k(\Omega_s(S)) \simeq \pi_k(\Omega_{F(s)}T) = \pi_{k+1}(T, F(s))$$

for all $s \in S$ and $k \geq 0$. ☐

In the following we follow the definitions of §1 in [20].

Definition 11.3.2. Given a morphism $f : S \to T$ of affine schemes we say that

- f is a *nilpotent embedding* if ${}^{\mathrm{c}\ell}f : {}^{\mathrm{c}\ell}S \to {}^{\mathrm{c}\ell}T$ is a closed embedding of classical affine schemes with nilpotent ideal of definition–let $W_{\mathrm{nil.}}$ denote the class of morphisms of affine schemes consisting of nilpotent embeddings;
- f is a *closed embedding* if ${}^{\mathrm{c}\ell}f : {}^{\mathrm{c}\ell}S \to {}^{\mathrm{c}\ell}T$ is a closed embedding of classical affine schemes–let $W_{\mathrm{cl.}}$ denote the class of morphisms of affine schemes consisting of nilpotent embeddings;
- f is a *reduced nilpotent embedding* if there exists $S_0 \to {}^{\mathrm{c}\ell}S$ a nilpotent embedding such that $S_0 \to T$ is a nilpotent embedding–let $W_{\mathrm{rnil.}}$ denote the class of morphisms of affine schemes consisting of reduced nilpotent embeddings;
- f is a *nil-isomorphism* if the induced morphism ${}^{\mathrm{red}}f : {}^{\mathrm{red}}S \to {}^{\mathrm{red}}T$ is an isomorphism–let $W_{\mathrm{nil-isom.}}$ denote the class of morphisms of affine schemes consisting of nil-isomorphisms;

[1]Notice that because $\Omega_s S$ is imposed connected, i.e. the requirement of the statement for $k = 0$, it doesn't matter which basepoint we take to calculate $\pi_k(\Omega_s(S))$ for $k \geq 1$.

- f is a *reduced closed embedding* if there exists $S_0 \to {}^{c\ell}S$ a nilpotent embedding such that $S_0 \to T$ is a closed embedding–let $W_{\mathrm{rcl.}}$ denote the class of morphisms of affine schemes consisting of reduced closed embeddings;
- f is *integral* if ${}^{c\ell}f : {}^{c\ell}S \to {}^{c\ell}T$ is an integral morphism of classical affine schemes–let $W_{\mathrm{int.}}$ denote the class of morphisms of affine schemes consisting of integral morphisms.

We notice the following inclusions

$$\begin{array}{ccc} W_{\mathrm{nil-isomorphism}} & & W_{\mathrm{int.}} \\ \subset\uparrow & & \uparrow\subset \\ W_{\mathrm{rnil.}} & \xrightarrow[\subset]{} & W_{\mathrm{rcl.}} \\ \subset\uparrow & & \uparrow\subset \\ W_{\mathrm{nil.}} & \xrightarrow[\subset]{} & W_{\mathrm{cl.}} \end{array}$$

We also need to introduce the following notion for derived rings.

Definition 11.3.3. A derived ring $R \in \mathrm{CAlg}$ is said to be an *Artinian derived ring* if

- $H^0(A)$ is Artinian;
- $H^i(A)$ is a finite $H^0(A)$-module for all $i \in \mathbb{Z}$; and
- $H^i(A) = 0$ for $i \ll 0$.

Moreover, A is said to be a local Artinian derived ring if A is also a local derived ring, i.e. $H^0(A)$ is a local discrete ring.

Let $\mathrm{Art}^{\mathrm{ft}}_{\mathrm{loc}}$ denote the subcategory of affine schemes of finite type generated by local Artinian derived rings, i.e. $R \in \mathrm{Art}^{\mathrm{ft}}_{\mathrm{loc}}$ if

a) $H^0(A)$ is a finite k-algebra[2];

b) $H^i(A)$ is a finite $H^0(A)$-module for all i (by (a) equivalently a finite k-module);

c) $H^i(A) = 0$ for $i \ll 0$.

We will also consider the subcategory $(\mathrm{Art}^{\mathrm{ft}}_{\mathrm{loc}})^{\mathrm{loc.}}$ consisting of morphisms in $\mathrm{Art}^{\mathrm{ft}}_{\mathrm{loc}}$ are maps $f : \mathrm{Spec}(B) = S \to T = \mathrm{Spec}(A)$ induced by morphisms of local derived rings, i.e. $\varphi : A \to B$ such that the induced morphism $H^0(\varphi) : H^0(A) \to H^0(B)$ takes the maximal ideal of $H^0(A)$ to a subset of the maximal ideal of $H^0(B)$.

Since one has a fully faithful embedding $\mathrm{Art}^{\mathrm{ft}}_{\mathrm{loc}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$ one can consider the classes of morphisms W_{Art} and $W_{\mathrm{Art-triv.}}$ where the first allows any morphisms between local Artininan affine scheme of finite type and the second requires that the induced morphisms between residue fields is an equivalence, in particular these are local morphisms.

We are interested in proving the analogues of Lemma B.2 and B.3 from [20].

Given a class $P \in \{W_{\mathrm{nil.}}, W_{\mathrm{cl.}}, W_{\mathrm{rnil.}}, W_{\mathrm{rcl.}}, W_{\mathrm{int.}}, W_{\mathrm{all}}\}$[3] a *P-nil square* is a pushout square

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle \jmath}\downarrow & & \downarrow \\ S' & \longrightarrow & T' \end{array} \tag{11.2}$$

where $f \in P$ and $\jmath$ is a nilpotent embedding.

[2] I.e. a finitely generated k-module.

[3] Here W_{all} just means all morphisms in the category of affine schemes.

Given a prestack $\mathscr{X}$ we say that $\mathscr{X}$ is *P-homogeneous* if for every square as (11.2) the following natural map

$$\mathscr{X}(T') \to \mathscr{X}(S') \underset{\mathscr{X}(S)}{\times} \mathscr{X}(T)$$

is an isomorphism.

For the purposes of bootstrapping homogeneity we need to consider a splitting of the condition above:

Definition 11.3.4. A prestack $\mathscr{X}$ is said to satisfy

(H_1^{P}) if for every square as (11.2) the natural map

$$\mathscr{X}(T') \to \mathscr{X}(S') \underset{\mathscr{X}(S)}{\times} \mathscr{X}(T)$$

is fully faithful;

(H_2^{P}) if for every square as (11.2) the natural map

$$\mathscr{X}(T') \to \mathscr{X}(S') \underset{\mathscr{X}(S)}{\times} \mathscr{X}(T)$$

is essentially surjective.

If $\mathscr{X}$ satisfies H_1^{P} and H_2^{P} we will say that $\mathscr{X}$ is *P-homogeneous.*

The following is a preliminary version of Lemma B.3.

Lemma 11.3.5. *Consider a $\mathscr{X}$ a prestack locally almost of finite type, such that $\mathscr{X}$ satisfies H_1^{P}, then the following are equivalent*

(1) $\mathscr{X}$ satisfies H_2^{P};

(2) $\mathscr{X}$ satisfies condition H_2^{P} for affine schemes in the subcategory ${}^{<\infty}Sch_{\mathrm{ft}}^{\mathrm{aff}}$;

(3) $\mathscr{X}$ satisfies H_2^{P} for diagrams

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle \jmath}\downarrow & & \downarrow \\ S' & \longrightarrow & T' \end{array} \tag{11.3}$$

where T is the Henselization of an affine scheme of finite type at a closed point, and f and $\jmath$ are finitely presented.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are tautological.

For (iii) $\Rightarrow$ (ii), consider a pushout diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{f} & S_2 \\ {\scriptstyle \jmath}\downarrow & & \downarrow \\ S_1' & \longrightarrow & S_2 \end{array}$$

where $\jmath$ is a nilpotent embedding and f belongs to the class P. Consider a closed point $s_2 \in S_2$ and let $T_2 := (S_2)^h_{s_2}$ be the Henselization of S_2 at the point s_2, i.e.

$$T_2 \simeq \lim_{U_2 \overset{p}{\to} S_2 \mid p \text{ is étale and } s_2 \in \mathrm{Im}({}^{c\ell}p)} U_2.$$

We first notice that the Henselization of S_2 lifts uniquely to a Henselization T_2' of S_2'. Indeed, by definition the morphism $f' : S_2 \to T_2$ is ind-étale, i.e. the filtered colimit of étale morphism, by Proposition 2.3.3 [10] the relative cotangent complex $T^*(S_2/T_2)$ vanishes. By [17, Proposition 5.5.3] any nilpotent embedding $S_2 \hookrightarrow S_2'$ is an iteration of square-zero extensions and since $\jmath$ is finitely presented there are only finitely many of them. In particular, we can assume that $S_2 \hookrightarrow S_2'$ is a square-zero extension, hence it is determined by a morphism

$$\alpha : T^*(S_2) \to \mathscr{F}$$

for some $\mathscr{F} \in \mathrm{QCoh}(S_2)^{\leq -1}$. Since the relative cotangent complex of f' vanishes one has an equivalence $T^*(T_2) \simeq (f')^*(T^*(S_2))$ which gives a morphism

$$(f')^*(\alpha) : T^*(T_2) \to (f')^*(\mathscr{F}),$$

that is a square-zero extension of T_2. Moreover, by the functoriality of square-zero extension we know that the following diagram commutes

$$\begin{array}{ccccc} S_1 & \xrightarrow{f'} & T_2 & \longrightarrow & S_2 \\ \downarrow{\scriptstyle \jmath} & & \downarrow & & \downarrow \\ S_1' & \longrightarrow & T_2' & \longrightarrow & S_2' \end{array}$$

Let $y \in \pi_0(\mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(S_2))$ from the diagram

$$\begin{array}{ccc} \mathscr{X}(S_2') & \longrightarrow & \mathscr{X}(T_2') \\ \downarrow & & \downarrow \\ \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(S_2) & \xrightarrow{\gamma} & \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(T_2) \end{array}$$

we get $\gamma(y) \in \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(T_2)$ and $\tilde{\gamma(y)} \in \mathscr{X}(T_2')$ a lift of $\gamma(y)$. Since $\mathscr{X}$ is locally almost of finite type, one has that

$$\mathrm{colim}_{U_2' \to S_2' \mid \text{étale}} \mathscr{X}(U_2') \xrightarrow{\simeq} \mathscr{X}(T_2'),$$

so we can assume that $\widetilde{\gamma(y)}$ is given by a some point $u_2' \in \mathscr{X}(U_2')$.

Finally, we claim that $u_2' \in \mathscr{X}(U_2')$ descends to a point of $x_2' \in \mathscr{X}(S_2')$. Since $\mathscr{X}$ is an étale sheaf we only need to show that there are isomorphism $\sigma : p_1^*(u_2') \simeq p_2^*(u_2')$ where $p_i^* : \mathscr{X}(U_2') \to \mathscr{X}(U_2' \underset{S_2'}{\times} U_2')$ (for $i = 1, 2$), and higher isomorphisms witnessing the compatibilities for σ.

Indeed, we now notice that the morphism $\mathscr{X}(U_2') \to \mathscr{X}(T_2') \to \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(T_2)$ factors as follows:

$$\begin{array}{ccccc} \mathscr{X}(S_2') & \longrightarrow & \mathscr{X}(U_2') & \longrightarrow & \mathscr{X}(T_2') \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(S_2) & \longrightarrow & \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(U_2) & \longrightarrow & \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(T_2) \end{array} \tag{11.4}$$

for some étale morphism $U_2 \to T_2$. Consider the diagram

$$\begin{array}{ccc} \mathscr{X}(U_2') & \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} & \mathscr{X}(U_2' \underset{S_2'}{\times} U_2') \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2} \\ \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(U_2') & \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} & \mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(U_2' \underset{S_2'}{\times} U_2') \end{array} \tag{11.5}$$

we obtain isomorphisms $\sigma' : p_1^*(\alpha(u_2')) \simeq p_2^*(\alpha(u_2'))$ since $\alpha(u_2')$ comes from a point in $\mathscr{X}(S_1') \underset{\mathscr{X}(S_1)}{\times} \mathscr{X}(S_2)$ and the commutativity of (11.5) gives $\alpha \circ p_1^*(u_2') \simeq \alpha \circ p_2^*(u_2')$. Finally, the assumption (H_1^{P}) gives that α_2 is fully faithful, so one obtains the isomorphism $\sigma : p_1^*(u_2') \simeq p_2^*(u_2')$.

$\square$

The notes Bootstrapping Homogeneity II also prove the following, which is an analogue of Lemma B.2 in [20].

Lemma 11.3.6. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of prestacks, the following are equivalent:*

(1) f satisfies H_1^{P};

(2) for every affine scheme $T \to \mathscr{X} \underset{\mathscr{Y}}{\times} \mathscr{X}$ the prestack $D_{\mathscr{X},T}$ defined by the following pullback diagram

$$\begin{array}{ccc} D_{\mathscr{X},T} & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathscr{X} & \xrightarrow{\Delta_{\mathscr{X}/\mathscr{Y}}} & \mathscr{X} \underset{\mathscr{Y}}{\times} \mathscr{X} \end{array}$$

is P-homogeneous.

In addition, if $\mathscr{X}, \mathscr{Y}$ are Zariski stacks locally almost of finite presentation and P is Zariski local, then these are equivalent to

(iii) condition (ii) where we only consider T locally almost of finite presentation over S.

In particular, given a prestack $\mathscr{X} \to S$ if $\Delta_{\mathscr{X}/S}$ is 0-geometric then $\mathscr{X}$ satisfies $(H_1^{\mathrm{W}_{\mathrm{all}}})$.

Proof. See note Bootstrapping Homogeneity II. $\square$

Chapter 12

Further topics

12.1 Ind-coherent sheaves and D-modules

12.1.1 Ind-coherent sheaves

The apparatus of ∞-categories allows one to develop sheaves theories in very general frameworks. Namely, quasi-coherent sheaves on a prestack $\mathscr{X}$ were defined as

$$\mathrm{QCoh}(\mathscr{X}) := \lim_{S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathscr{X}}} \mathrm{QCoh}(S)$$

with respect to $*$-pullback.

Similarly, the theory of ind-coherent sheaves, defined for an scheme X as

$$\mathrm{IndCoh}(X) := \mathrm{Ind}(\mathrm{Coh}(X))$$

can be extended to a large class prestacks, namely all those locally almost of finite type, with arbitrary pullbacks and ind-inf-schematic pushforwards. Moreover, the use of derived geometry allows one to make sense of base change formulas for ind-proper morphisms.

The main problem solved by ind-coherent sheaves is to provide for any morphism $f : X \to Y$ a pullback functor

$$f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$$

which preserves infinite direct sums. At first glance the theory of ind-coherent sheaves doesn't seem to change much, essentially for any scheme X one has an equivalence:

$$\mathrm{IndCoh}(X)^{\geq -n} \simeq \mathrm{QCoh}(X)^{\geq -n}$$

for any $n \in \mathbb{N}$. So it seems to only be relevant to homological algebra questions which are sensitive to objects with arbitrarily negative cohomology. However, the pushforward in the generality that it is constructed encodes a non-trivial operation as the next section vaguely describe.

12.1.2 D-modules

One of the main applications of ind-coherent sheaves is to construct D-modules on laft prestacks. Given a prestack locally almost of finite type $\mathscr{X}$ we let

$$\begin{aligned} \mathscr{X}_{dR} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} &\to \mathrm{Spc} \\ S &\mapsto \mathscr{X}({}^{\mathrm{red}}S), \end{aligned}$$

where ${}^{\mathrm{red}}S \hookrightarrow S$ is the reduced classical affine scheme of the underlying classical affine scheme $c\ell$S. It is a bit hard to give an intuition of what the functor $\mathscr{X}_{dR}$ is doing.

In [17]*Chapter 8 a theory of Lie algebroids on laft prestacks with deformation theory is developed. By definition Lie algebroids on $\mathscr{X}$ are groupoid objects in the category $(\mathrm{PreStk})_{\mathrm{nil-isom}}$ whose 0th object is $\mathscr{X}$. Any such object describes a prestack $\mathscr{Y}$ under $\mathscr{X}$ such that[1]

$$f:\mathscr{X}\to\mathscr{Y}\ \text{ is a nil-isomorphism, i.e. }\ {}^{\mathrm{red}}f:{}^{\mathrm{red}}\mathscr{X}\xrightarrow{\simeq}{}^{\mathrm{red}}\mathscr{Y}.$$

The main point is that the canonical projection $\mathscr{X}\to\mathscr{X}_{dR}$ corresponds to the (formal) groupoid given by

$$\cdots\to(\mathscr{X}\underset{\mathrm{pt}^{\wedge}_{\mathscr{X}}}{\times}\mathscr{X}\underset{\mathrm{pt}^{\wedge}_{\mathscr{X}}}{\times}\mathscr{X})\to(\mathscr{X}\times\mathscr{X})^{\wedge}_{\mathscr{X}}\to\mathscr{X}$$

since

$$(\mathscr{X}\times\mathscr{X})^{\wedge}_{\mathscr{X}}\simeq\mathscr{X}\underset{\mathrm{pt}^{\wedge}_{\mathscr{X}}}{\times}\mathscr{X}$$

where $\mathrm{pt}^{\wedge}_{\mathscr{X}}$ is the formal completion of the point with respect to $\mathscr{X}$, i.e.[2]

$$\mathrm{pt}^{\wedge}_{\mathscr{X}}:=\mathscr{X}_{dR}\underset{\mathrm{pt}_{dR}}{\times}\mathrm{pt}.$$

In other words, the fiber of $\mathscr{X}_{dR}\to\mathrm{pt}_{dR}$ is isomorphism to the formal completion of $\mathscr{X}$ along the diagonal.

Finally, the theory takes off the ground with the definition:

$$D-\mathrm{mod}(\mathscr{X}):=\mathrm{IndCoh}(\mathscr{X}_{dR}),$$

which realizes Grothendieck's point of view on differential operators as crystals in a very vast generality.

12.2 Derived analogues of usual geometric conditions

12.2.1 Local complete intersection

In [Stacks, §37.56] the notion of a lci morphism between classical schemes is defined, i.e. $f:X_0\to Y_0$ locally on X_0 can be factored as a regular embedding followed by a smooth morphism.

One has a notion for derived schemes, $f:X\to Y$ is a lci morphism if the relative cotangent complex $T^*(X/Y)$ has Tor-amplitude in $[-1,0]$. The following can be deduced from [41, Lemma 2.4]

Proposition 12.2.1. *A morphism $f:X\to Y$ between schemes is lci if and only if ${}^{c\ell}f:{}^{c\ell}X\to{}^{c\ell}Y$ is lci.*

Proof. The statement is local on X, so we can reduce $X=\mathrm{Spec}(B)$. Since being a closed embedding and smooth is local on the target we can reduce to $Y=\mathrm{Spec}(A)$. Finally, for B a discrete algebra we notice that by [Stacks, Tag 0654] one has $\mathbb{L}_{B/A}$ has Tor-amplitude in $[-1,0]$ if an only if $\mathbb{L}\in\mathrm{Mod}_B^{\geq-1,\leq0}$. The last condition is equivalent to lci by the deep result [4, Theorem 1.2]. □

Another interesting reference is [53] where Toën proves the following

Theorem 12.2.2. *Let $f:X_0\to Y_0$ be a proper and lci morphism between classical schemes, then*

$$f_*(\mathit{Perf}(X_0))\subseteq\mathit{Perf}(Y_0).$$

The proof uses the equivalence of Proposition 12.2.1 and base change for *derived schemes.*

[1]Here ${}^{\mathrm{red}}\mathscr{X}$ denotes the restriction of $\mathscr{X}$ to the category $({}^{c\ell}\mathrm{Sch}^{\mathrm{aff,red}})^{\mathrm{op}}$ of reduced classical affine schemes.

[2]In this flexible framework of prestacks, one can take the formal completion with respect to arbitrary maps. For instance, $p:\mathscr{X}\to\mathrm{pt}$ which, in general, is very far from a closed embedding.

12.2.2 Derived blow up

In [23] the authors prove the following impressive result, which settles Weibel's conjecture.

Theorem 12.2.3. *For any Noetherian classical scheme X of dimension d, one has $K_{-i}(X) = 0$ for all $i > d$.*

The proof makes essential use of the following technical result. Let $Y \to X$ be a regular closed embedding of classical and Y_n denote the nth infinitesimal neighborhood of Y in X. For each $n \geq 1$ consider the following abstract blow up diagram

$$\begin{array}{ccc} E_n & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & X, \end{array} \tag{12.1}$$

i.e. $\tilde{X} \to X$ is proper and $\tilde{X}\backslash E_n \overset{\simeq}{\to} X\backslash Y_n$ is an isomorphism. Then one has

Theorem 12.2.4. *The associated diagram obtained by applying algebraic K-theory to (12.1)*

$$\begin{array}{ccc} K(X) & \longrightarrow & lim_{n\geq 1} K(Y_n) \\ \downarrow & & \downarrow \\ K(\tilde{X}) & \longrightarrow & lim_{n\geq 1} K(E_n) \end{array} \tag{12.2}$$

is a pullback diagram in the ∞-category of pro-spectra.

We make two comments about the above excision type of theorem. First, it is *not* true that any abstract blow up diagram, i.e. a non-regular closed immersion, gives rise to a Cartesian diagram in algebraic K-theory. Second, even though the statement of the theorem doesn't mention any derived schemes, the proof actually works by introducing the notion of derived blow ups to extend Thomason's result that one has Cartesian diagrams in K-theory for regular closed embeddings to deduce the result for the limit objects as in (12.2).

The paper [24] studies derived blow ups more systematically, including giving a more intrinsic definition.

12.2.3 Results about classical stacks

Daniel Halpern-Leinster has many interesting papers which use derived geometry and ∞-categorical methods to obtain new results for objects from classical algebraic geometry. For instance Theorem C of [] is a great example of that. Informally it states that for a classical stack $\mathscr{X}_0$ locally almost of finite presentation and with affine diagonal a Θ-stratum $\mathscr{S}_0 \hookrightarrow \mathscr{X}_0$ one obtains a semi-orthogonal decomposition type of statement for $\mathrm{h}\,\mathrm{QCoh}(\mathscr{X}_0)$ with one of its parts identified with $\mathrm{h}\,\mathrm{QCoh}(\mathscr{X}_0\backslash\mathscr{S}_0)$.

12.3 Loop spaces

12.3.1 D-modules as sheaves on the loop space

We follow [54, §4.4] to give an idea of what one can do with the derived space. For any stack $\mathscr{X}$ consider

$$\mathscr{L}\mathscr{X} := \mathrm{Maps}_{\mathrm{Stk}}(\underline{S^1}, \mathscr{X})$$

where $\underline{S^1}$ is the stack associated to the constant prestack that sends any affine scheme S to the topological space S^1 seem as object of Spc. Notice that in the ∞-category one has an equivalence

$$S^1 \simeq [0,1] \underset{\mathrm{pt}\sqcup\mathrm{pt}}{\sqcup} [0,1] \simeq \mathrm{pt} \sqcup_{\mathrm{pt}\sqcup\mathrm{pt}} \mathrm{pt},$$

where $\mathrm{pt} \to \{0\}$ and $\mathrm{pt} \to \{1\}$. Since we can compute the mapping stack by consider the sheafification of the mapping prestacks, one has

$$\mathscr{L}\mathscr{X} \simeq \mathrm{L}\left(\mathrm{Maps}_{\mathrm{PreStk}}(\underline{\mathrm{pt} \sqcup_{\mathrm{pt}\sqcup\mathrm{pt}} \mathrm{pt}}), \mathscr{X}\right) \simeq \mathrm{L}\left(\mathscr{X} \underset{\mathscr{X}\times\mathscr{X}}{\times} \mathscr{X}\right).$$

Let's compute this in a couple of cases.

Example 12.3.1. (i) $\mathscr{X} = S = \mathrm{Spec}(A)$ for some $A \in \mathrm{CAlg}$ one obtains

$$\mathscr{L}\mathrm{Spec}(A) \simeq \mathrm{Spec}(A \underset{A\otimes_k A}{\otimes} A);$$

(ii) $\mathscr{X} = Z$ a derived scheme, one has a canonical map

$$\pi_Z : \mathscr{L}Z \to Z$$

obtained from either of the projections. We claim that for any $z : S \to Z$ one has $\pi_Z^{-1}(z) \simeq \mathrm{Spec}_S(\mathrm{Sym}^\bullet(T_z^* Z[1]))$, that is the map π_Z is affine with fiber isomorphic to the relative spectrum $\mathrm{Spec}_Z(\mathrm{Sym}^\bullet(T^*X[1]))$;

(iii) For $\mathscr{X} = \mathscr{X}_0$ a classical 1-truncated stack, one has $\mathscr{L}\mathscr{X}_0$ is a derived enhancement of the inertia stack $\mathscr{I}\mathscr{X}_0$, i.e.

$$^{c\ell}\mathscr{L}\mathscr{X}_0 \simeq \mathscr{I}\mathscr{X}_0;$$

(iv) for $\mathscr{X} \simeq X/G$ where X is a scheme and G a group scheme one has

$$\mathscr{L}(X/G) \simeq X^{hG}/G$$

where X^{hG} is defined by the fiber product

$$X^{hG} := (X \times G) \underset{X\times X}{\times} X.$$

In particular, one has $\mathscr{L}G \simeq G/G$, where G acts by conjugation.

One observes that

$$\Gamma(\mathscr{L}Z, \mathscr{O}_{\mathscr{L}Z}) \simeq \Gamma(Z, \mathrm{Sym}_{\mathscr{O}_Z}(T^*Z[1])) \simeq \oplus_{p\geq 0}\Gamma(Z, (\bigwedge^p TZ)[p]).$$

This starts to suggest that sheaves on the space $\mathscr{L}Z$ are closely related to D-modules on the space Z. For one to make that precise one needs to understand what plays the whole of the differential in the algebra $\mathrm{Sym}_{\mathscr{O}_Z}(T^*Z[1])$–it turns out that it the rotation of the loop in $\mathscr{L}Z$.

First one has a Koszul duality type of statement that says the following:

Proposition 12.3.2. *One has equivalence of categories:*

(i) $Fun(BS^1, Mod_k) = Mod_k^{S^1} \simeq Mod_{(H^\bullet(S^1))^\vee} = Mod_{k[\epsilon]}$, *where* $|\epsilon| = -1$

(ii) $CAlg^{S^1} = CAlg(Mod_k^{S^1}) \overset{\varphi}{\to} CAlg(Mod_{k[\epsilon]}) = CAlg_{k[\epsilon]/}$.

The category in (i) is refereed to as the category of mixed complexes and the category in (ii) as mixed commutative algebras. Moreover, for any $A \in CAlg$ the underlying k-module of $\varphi(S^1 \otimes A)$ is $Sym_A(\mathbb{L}_A[1])$. One then writes $DR(A) = \varphi(A)$ and says this is the de Rham mixed commutative algebra associated to A.

The above result can be put in families to yield:

Theorem 12.3.3. *For any derived scheme Z one has an equivalence*

$$\varphi(\mathscr{O}_{\mathscr{L}Z}) \simeq DR(\mathscr{O}_Z)$$

where $DR(\mathscr{O}_Z)$ is defined by using some local description of $DR(A)$, normally relying on the proposition above. When one considers sheaves this gives an equivalence of categories

$$Coh^{S^1}(\mathscr{L}Z)[\beta^{-1}] \simeq D-mod^{\mathbb{Z}/2}(Z),$$

here the category on the right are S^1-equivariant sheaves on $\mathscr{L}Z$ and the β and $\mathbb{Z}/2$ are there to make these categories 2-periodic.

There are versions of this theorem where one extends Z to certain derived stacks and obtain a more refined statement than just an equivalence of the 2-periodic part of the categories.

12.4 Symplectic geometry

12.4.1 Shifted symplectic structures

We will explain the idea in the affine case. We will need the category of *graded mixed complexes* over k, that can be simply defined as

$$\mathrm{Mod}_k^{\mathrm{gr.,m.}} := \mathrm{Mod}(B(\mathbb{G}_{\mathrm{m}} \rtimes \mathbb{G}_{\mathrm{a}}))$$

where $\mathbb{G}_{\mathrm{m}}$ acts on $\mathbb{G}_{\mathrm{a}}$ by scaling with weight 1. This category has a functor to *graded complexes*

$$NC^{\mathrm{w}} : \mathrm{Mod}_k^{\mathrm{gr.,m.}} \to \mathrm{Mod}_k^{\mathrm{gr.}}$$

given by π_* where $\pi : B(\mathbb{G}_{\mathrm{m}} \rtimes \mathbb{G}_{\mathrm{a}}) \to B(\mathbb{G}_{\mathrm{m}})$ is induced by the projection to the $\mathbb{G}_{\mathrm{m}}$ factor.

Given $A \in \mathrm{CAlg}$ another characterization of $\mathrm{DR}(A)$ without using the Proposition from the previous section is as follows

Definition 12.4.1. There is an unique object $\mathrm{DR}(A) \in \mathrm{CAlg}(\mathrm{Mod}_k^{\mathrm{gr.,m.}})$ whose underlying graded algebra satisfies

$$\mathrm{DR}(A) \simeq \mathrm{Sym}^*(\mathbb{L}_A[1])$$

where $\mathrm{Sym}^p(\mathbb{L}_A[1])$ is part of degree p. Moreover, the space of closed p-forms is defined as

$$\Omega_A^{p,\mathrm{cl}} := NC^{\mathrm{w}}(\tau_{\geq p}(DR(A))),$$

where $\tau_{\geq p}(DR(A))$ denotes the truncation of $DR(A)$ to its degree greater than or equal to p part as an object of $\mathrm{Mod}_k^{\mathrm{gr.}}$.

Concretely, one can write

$$\Omega_A^{p,\mathrm{cl}} \simeq \oplus_{n\geq 0} \wedge^{n+p} \mathbb{L}_A[-p]$$

from which it is clear that there is a map

$$\Omega_A^{p,\mathrm{cl}} \to \Omega_A^p := \wedge^p \mathbb{L}_A[-p].$$

However, there is no canonical section–for a p-form to be closed involves extra data, i.e. an element of $\Omega_A^{p,\mathrm{cl}}$ projecting to it.

Finally, one has a definition

Definition 12.4.2. For any commutative algebra A a *n-shifted symplectic structure* on $\mathrm{Spec}(A)$ is a 2-form of degree n, i.e. a map

$$\omega : k[-n] \to \Omega_A^2$$

such that

a. ω induces an isomorphism
$$\mathbb{L}_A \xrightarrow{\simeq} \mathbb{L}_A^\vee[-n];$$

b. ω is closed, i.e. there exists an element $\widetilde{\omega} : k[-n] \to \Omega_A^{2,\mathrm{cl}}$ which projects to ω.

As we mentioned before this definition can be extended to a large class of derived stacks, i.e. any admitting a smooth cover by representable affine schemes. Many aspects of this theory have been developed, maybe we roughly only state the AKSZ result which realizes some intuitive physical picture:

Theorem 12.4.3. *Given a derived stack X with an orientation of degree d and Y an n-shifted derived stack, then the mapping stack $Maps(X, Y)$ has an $(n - d)$-shifted symplectic structure.*

The reader is refereed to §2.1 from [40] for more details in the above theorem, including a discussion of orientation in a derived stack.

Chapter 13

Summary

In this last chapter we summarize what we covered in each lecture and point to resources in the literature which were useful to the author while preparing. We have no pretense of being exhaustive here and we apologize in advance to anyone whom we left out.

13.1 General breakdown

Talks 1-2: General introduction to what is derived algebraic geometry. There are many good surveys of this material. I would recommend: Chapter 0 of [35], Toën's review [54], there are also a couple of online lecture notes (see).

Talks 3-8: introduced most of the technology of ∞-categories that we used in the course. The introduction of the basics of ∞-categories using the quasi-categories as a model is in [33, §1]. Other useful resources are: [38] for an informal discussion of why ∞-categories, [] The material of talks 4 and 5 can be found in §1.1-1.3, 2.4, 2.5, and 4.4 in [34]. For talks 6 and 7 see §4.3 [34] and Part 5 of [44] for under-categories and [16, Chapter 1, §1.3], [33, Chapter 2], §4.2 [34] and [37] for discussions about fibrations and the straightening and unstraightening equivalence. For the construction of the Yoneda functor by a clever straightening/unstraightening procedure see [16, Chapter 1, §1.5]. For Talk 8 see [32, Chapter 1] for an extensive treatment of stable categories.

Talks 9-15: discussed the affine aspect of the theory, in other words: derived commutative algebra. The material of Talks 9, 10 and 12 is spread in different Chapters of [32] and often formulated in the more general language of ∞-operads. An good initial discussion is [16, Chapter 1, §3, 6 and 8]. For most proofs the reader can collect them throughout Chapter 2, §1.2-3, §4.1-2, Chapter 3, §3.3, §4.1, Chapter 4, §5 and Chapter 7, §1 of [32]. For Talk 11 a good reference is §4.1 [27]. Talk 13 summarizes the technical results of §4.4.2 and §4.5.4 of [32]. For Talk 14 see Chapter 7, §2.2-3 of [32].

Talks 16-24: we actually did some (non-affine) geometry. For Talk 16 a quick introduction for the topologies on affine schemes is [16, Chapter 2, §2.1]. A detailed treatment can be found in §7.5 [32] and Appendix B of [35]. We proved descent for affine schemes following §1.3.2 of [56]. Talk 17 followed pretty closely Chapter 2, §3 [16]. For Talk 18 we followed §7.3 of [32] with an emphasizes on §7.3.3 Find a reference for the base change result. for the affine case and Chapter 1, §1-4 of [17] for the global case. For Talk 19 the discussion of the important connectivity estimates is in §7.4.3 of [32] see also the more informal Chapter 1, §5 of [17]. For Talk 20 we followed the proofs in §2.2.2 of [56]. Talk 21 followed Chapter 1, §6-9 of [17]. In Talk 22 our definition of geometric stacks is following §5.1 of [26], the rest of the discussion of results follows Chapter 2, §4 of [16]. Talk 23 follows very closely the paper []. Talk 24 is a summary of §7 of [26], but see also Chapter 17, §3 and Chapter 18, §3 of [35] for relevant but somewhat slightly different material.

13.2 Talk by talk summary

Talks:

1. **What is DAG?** We talked about the functor of points perspective on classical algebraic geometry (see [] for a classical exposition). Then gave an intuition on how to generalize the two categories involved in the definition of a functor of points: commutative algebras and sets.

2. **Why DAG?** Formal results as base change hold with less assumptions, application to geometric representation theory ([45, 9]), formalization of the notion of obstruction theory ([]), and fundamental classes ([25]). **Tools of DAG?** Cotangent complex and mentioned facts about it (see §7).

3. **Why ∞-categories?** Example of glueing derived categories. **Definitions:** quasi-categories, Kan complexes, homotopy between morphisms and homotopy category. **Facts:** the space of compositions is contractible, the homotopy $\mathrm{h}\mathscr{C}$ associated to an ∞-category $\mathscr{C}$ only depends on its 2-skeleton as a simplicial set.

4. **Definitions:** isomorphisms in an ∞-category , mapping space. **Facts:** proved that composition is homotopy associative. **Remarks:** homotopy category of a topological category and a word on models.

5. **Definitions:** homotopy coherent nerve, dg nerve, the ∞-category of spaces and of ∞-categories. Defined initial and final objects in an ∞-category. Discussed how to make sense of limits and colimits: settled with the definition using undercategories.

6. Defined undercategories. Motivation for fibrations by mentioning adjoint functors and Yoneda embedding. Defined p-Cartesian morphisms and Cartesian fibrations. Formulate the straightening/unstraightening equivalence of categories.

7. Mentioned the particular case of Cartesian fibrations in spaces (i.e. right fibrations) and compatibility of straightening and unstraightening results. Commented on the opposite case, i.e. coCartesian and left fibrations. **Examples:** slice category, Yoneda functor (using St and Un). Discussion of limits and colimits in Spc and Cat_∞.

8. Introduced stable ∞-categories and listed their properties. Computation of the cofiber functor. Constructed the ∞-category of spectra and the derived category of a commutative ring k.

9. Introduced the notion of symmetric monoidal ∞-categories. **Examples:** Cat_∞, Spc, Pr^{L}, and $\mathrm{Cat}_\infty^{\mathrm{st}}$. Defined the notion of commutative algebra object and discussed why a morphism of ∞-operads only preserves inert (Cartesian) morphisms.

10. Cleaned up the discussion from last time on CAlg-objects, introduced the notion of n-truncated and discrete objects in an oo-cat, recalled t-structures, n-truncated objects in $\mathscr{C}^{\leq 0}$ are $\mathscr{C}^{\geq -n, \leq 0}$, Vect has a t-structure with heart the ordinary category of vector space, is symmetric monoidal, the $\otimes$-structure restricts to $\mathrm{Vect}^{\leq 0}$, (nonstrict) cdgas are defined as $\mathrm{CAlg}(\mathrm{Vect}^{\leq 0})$, the discrete objects in this category are $\mathrm{CAlg}(\mathrm{Vect}^{\heartsuit})$, i.e. ordinary commutative algebras.

11. Introduced $\mathbb{E}_\infty$-algebras and stated how they are equivalent to (non-strict) cdgas. Defined sifted ∞-categories and simplicial commutative rings as the sifted completion of f.g. free k-algebras. Stated the equivalence of the sifted completion with the strict model of simplicial commutative rings. Defined monoidal ∞-categories and functor from Δ^{op} to Fin_*.

12. Introduced Δ^+ category and the notion of a module category. **Example:** symmetric monoidal ∞-categories as a module over themselves, e.g. Spctr, Mod_A for A a derived ring. Stated Schwede–Schipley's theorem on recognition of module categories.

13. Construction of a symmetric monoidal ∞-category from a symmetric monoidal model category. **Examples:** dg k-modules and cdgas. Stated equivalence of strict cdgas and non-strict cdgas and of the tensor structure of the derived category with that of the module category over a derived ring. Defined subcategories of R-modules for R a derived ring.

14. **Definitions:** flat, projective, perfect and almost perfect R-modules. Characterization of these properties with some of the ideas of the proofs.

15. More about R-modules. Defined vector bundles over R and their equivalent characterizations. Introduced prestacks and natural conditions on them: n-coconnective, convergence, laft and k-truncated.

16. Introduced topologies on affine schemes, descent conditions, and defined stacks. Stated flat descent for R-modules and its consequences. Sketch of two possible proofs. Sanity check: affine schemes are stacks, i.e. satisfy étale descent. **Example:** for $R = \mathrm{Sym}(V)$ where V is a perfect connective k-module, one has that $\mathrm{Spec}(R)$ is a 0-coconnective prestack. **Remark:** characterization of compact and projective objects of CAlg.

17. Introduced derived schemes (following [16, Chapter 2, §3]). Discussion of Zariski atlas, étale surjection and presentation of a derived scheme as a geometric realization of its Cečh cover. **Remark:** groupoid objects in ∞-categories. **Example:** projective space over $k[\eta]$, $|\eta| = -2$. A scheme is affine iff its underlying classical scheme is affine. Proved that the restriction of the definition of derived schemes to discrete algebras produces 0-truncated objects and recovers the usual notion of separated classiccal schemes. Discussed the notion of an n-coconnective stack (this is in the notes to Lecture 16).

18. Introduced the cotangent complex. First with a discussion of the affine situation motivated by the classical notion of Kähler differentials. **Remark:** spectrum objects in $\mathrm{CAlg}_{/A}$ are equivalent to A-modules for any derived ring A. Use this equivalence to construct square-zero extensions. Definition of a cotangent complex for a prestack, including a mention of the relative situation. Specialized to the case $A \to B$ a map of commutative algebras and computed that $\mathbb{L}_{B/A} \simeq B \otimes_A V$ for $B = \mathrm{Sym}_A(V)$.

19. **Facts:** base change for cotangent complex and fiber sequence results. Introduced the terminology about n-connective maps and stated the connectivity estimate result, an idea of its proof and some consequences.

20. **Definitions:** formally smooth and formally étale morphisms. Proved that formally étale and finitely presented is equivalent to étale. **Remark:** relation between cotangent complex of a derived extension of a classical scheme and the usual cotangent complex as defined by Illusie. Defined infinitesimally cohesive and mentioned result about factorizing a nilpotent embedding as a sequence of square-zero extensions.

21. Defined deformation theory. **Example:** schemes admits deformation theory. Three consequences of deformation theory: (i) descent at classical level implies full descent; (ii) isomorphism is detected at the classical level plus isomorphic cotangent complexes, and (iii) cotangent complex criteria for laft.

22. Defined geometric stacks. **Facts:** (i) n-geometric implies the underlying classical prestack is n-truncated and that its cotangent complex is concentrated in cohomological degree $\leq n$, (ii) quotient of a groupoid object in $(n-1)$-geometric stacks with $(n-1)$-geometric morphisms is n-geometric. **Examples:** BG for G a group scheme is 1-geometric, Vect is 1-geometric.

23. Proof that the stack of perfect complexes is geometric following [].

24. Formulated Lurie's representability theorem for n-geometric stacks following [26, §7]. Remarks about how this result relates to Artin's and to Pridham's result. Idea of proof.

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