∞-geometry and ∞-categories in representation theory

From finite groups of Lie type to p-adic groups

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Aim of the talk

How do elementary questions about representations lead to geometry and to ∞-categories?

1. Compute characters by hand for S_3 and $\mathrm{GL}_2(\mathbb{F}_q)$.
2. Use geometry and categories to systematically understand groups such as $\mathrm{GL}_n(\mathbb{F}_q)$.
3. Replace matrices over a field by matrices of Laurent series.
4. See how infinite-dimensional geometry enters the representation theory of $\mathrm{GL}_n(\mathbb{Q}_p)$, and where our work contributes.
5. Mention related geometric ideas at SIMIS.

Outline

Representation theory of finite groups (of Lie type)

A categorical reformulation

Some ∞-dimensional geometry

p-adic representations and characters

Other related work at SIMIS

Representation theory of finite groups (of Lie type)

Motivating questions

Let Γ be a finite group. A finite-dimensional complex representation is a homomorphism

$$\rho : \Gamma \longrightarrow \mathsf{GL}(V).$$

Its character is the function $\chi_V(g) = \mathsf{Tr}(\rho(g) \mid V)$.

1. What are the irreducible representations?
2. What are their characters?
3. How do we compute their values?

Since $\chi_V(h^{-1}gh) = \chi_V(g)$, it is enough to consider conjugacy classes.

S_3: why three irreducible representations?

Over $\mathbb{C}$, every finite-dimensional representation of a finite group is a direct sum of irreducibles, and there are as many irreducibles as conjugacy classes.

The two key facts are:

- averaging a projection gives an invariant complement;
- the center of $\mathbb{C}[\Gamma]$ has one basis indexed by conjugacy classes, and another indexed by irreducible representations.

For S_3, the conjugacy classes are $\{e\}, \{(12), (13), (23)\}, \{(123), (132)\}$.

We deduce that S_3 has exactly three irreducible complex representations.

S_3: the representations and their characters

Three irreducibles, with $\sum_V (\dim V)^2 = 6$: their dimensions are $1, 1, 2$. They are $\mathbf{1}$, sgn, and the sum-zero plane

$$V_{\mathrm{std}} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\}.$$

Since $\mathbb{C}^3 = \mathbf{1} \oplus V_{\mathrm{std}}$, $\chi_{\mathrm{std}}(g) = \#\{\text{letters fixed by } g\} - 1$.

	e	transpositions	3-cycles
class size	1	3	2
$\chi_{\mathbf{1}}$	1	1	1
χ_{sgn}	1	-1	1
χ_{std}	2	0	-1

The same question for finite groups of Lie type

Now consider finite groups such as

$$\mathrm{GL}_n(\mathbb{F}_q), \qquad \mathrm{SL}_n(\mathbb{F}_q), \qquad \mathrm{Sp}_{2n}(\mathbb{F}_q).$$

Write $q = p^r$ and $k = \overline{\mathbb{F}}_q$. These groups have the form $G = \mathbf{G}^F$, where $\mathbf{G}$ is a connected reductive algebraic group over k and F is a Frobenius morphism. For $\mathbf{G} = \mathrm{GL}_{n,k}$, we use $F((a_{ij})) = (a_{ij}^q)$.

The defining representation on k^n supplies the geometry of flags. Deligne–Lusztig use flags in a prescribed relative position with their Frobenius transforms; cohomology produces characteristic-zero representations.

For GL_2, flags are simply lines.

The split GL_2 example

Take $\mathbf{G} = \mathbf{GL}_{2,k}$, with

$$F((a_{ij})) = (a_{ij}^q), \qquad G = \mathbf{GL}_2(\mathbb{F}_q), \qquad \text{for } q = p^r.$$

Let $\mathbf{B}$ be the upper triangular subgroup and $\mathbf{T}$ the diagonal torus. Then

$$X = \mathbf{G}/\mathbf{B} \simeq \mathbb{P}^1_k, \qquad W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} = \{1, s\} \simeq S_2.$$

A line $L \subset k^2$ determines its stabilizer $\mathbf{B}_L$.

Two lines have relative position 1 if they coincide, and s if they are distinct. Also $F([x : y]) = [x^q : y^q]$.

For $w \in W$, choose a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ and set

$$X_w = \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} : g^{-1}F(g) \in \mathbf{B}\dot{w}\mathbf{B}\}.$$

Equivalently, X_w parametrizes Borels $\mathbf{B}'$ for which $(\mathbf{B}', F(\mathbf{B}'))$ has relative position w.

For $\mathbf{GL}_2$ this becomes

$$X_1 = \mathbb{P}^1(\mathbb{F}_q), \qquad X_s = \mathbb{P}^1_k \setminus \mathbb{P}^1(\mathbb{F}_q).$$

The group G acts on these varieties, and therefore on their compactly supported cohomology.

Deligne–Lusztig induction

With coefficients $\overline{\mathbb{Q}}_\ell$ ($\ell \neq p$), the G-action on X_w gives a virtual representation

$$R_w(1) = \sum_i (-1)^i [H^i_c(X_w, \overline{\mathbb{Q}}_\ell)].$$

Its irreducible constituents, as w varies, are called *unipotent representations*.

To reach all irreducibles, use a finite cover $Y_w \to X_w$ with commuting left G-action and right finite-torus action. Take torus-character summands of its cohomology to obtain $R_w(\theta)$.

Deligne–Lusztig. Every irreducible occurs with nonzero coefficient in some $R_w(\theta)$. These are virtual representations, so coefficients can be negative.

The two varieties give two virtual representations

For $\theta = 1$, the local system is constant. The permutation representation on rational lines decomposes as

$$\overline{\mathbb{Q}}_\ell[\mathbb{P}^1(\mathbb{F}_q)] = \mathbf{1} \oplus \mathrm{St}, \qquad \dim \mathrm{St} = q.$$

Consequently,

$$R_1(1) = \mathbf{1} + \mathrm{St}, \qquad R_s(1) = \mathbf{1} - \mathrm{St}\,.$$

The second identity follows by removing the $q+1$ rational points from $\mathbb{P}^1_k$ and using compactly supported cohomology.

Already in this example the minus sign records a cohomological degree.

Character values: what has to be computed?

Every $g \in G$ has a commuting decomposition $g = su$, with s semisimple and u unipotent.

The character formula combines torus character values at s with *Green functions*: cohomological character values at u in the connected centralizer of s.

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ (a \neq b) : \ u = 1, \qquad a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \ s = aI.$$

For GL_2, we can see these different behaviors simply by counting invariant rational lines. The next slide does the calculation.

The calculation for $\mathrm{GL}_2(\mathbb{F}_q)$

Write $R_w(1)(g)$ for the character value of $R_w(1)$ at g. A permutation trace counts fixed basis vectors, so

$$R_1(1)(g) = \#\{L \in \mathbb{P}^1(\mathbb{F}_q) : gL = L\}, \qquad \mathrm{St}(g) = R_1(1)(g) - 1.$$

type of g	$R_1(1)(g)$	$\mathrm{St}(g)$	$R_s(1)(g)$
scalar aI	$q+1$	q	$1-q$
distinct eigenvalues in $\mathbb{F}_q$	2	1	0
nontrivial Jordan block	1	0	1
irreducible quadratic characteristic polynomial	0	-1	2

In particular, the identity and a nontrivial unipotent element have different character values.

What remains after Deligne–Lusztig induction?

We have constructed virtual representations, and every irreducible occurs in some $R_w(\theta)$.

Two problems remain:

1. Decompose the virtual representations into irreducibles.
2. Compute their character values; the character formula brings us to Green functions in centralizers.

For GL_2, counting lines solved our example. For general groups, character sheaves organize families of cohomologies and provide a geometric basis to compare with irreducible characters.

A sheaf-theoretic turn: families over the group

- **The idea.** Let the group element vary: assemble the cohomology of flag varieties into a sheaf over $\mathbf{G}$.
- **The example.** For GL_2, use the family of invariant lines:

$$\widetilde{\mathbf{G}} = \{(g, L) \in \mathbf{G} \times \mathbb{P}^1 : gL = L\} \xrightarrow{\pi} \mathbf{G}, \qquad (g, L) \longmapsto g.$$

- **Grothendieck's dictionary.** Alternating Frobenius traces on the stalks recover our invariant-line character $R_1(1)$.
- **Lusztig's payoff.** Character sheaves are the geometric building blocks. The normalized trace functions of the F-stable ones form a basis of class functions on $G = \mathbf{G}^F$.

Comparison. Lusztig–Shoji link these functions to Deligne–Lusztig theory via *almost characters* (with suitable hypotheses and normalizations).

A categorical reformulation

Monoidal categories: composing operations on flags

Let $S = \mathbb{P}^1(\mathbb{F}_q)$, the set of flags for $\mathbf{GL}_2$. The finite Hecke algebra consists of *invariant kernels*

$$H = \mathbf{Fun}(S \times S, \overline{\mathbb{Q}}_\ell)^G, \qquad (a * b)(F_1, F_3) = \sum_{F_2 \in S} a(F_1, F_2) b(F_2, F_3).$$

There are two G-orbits on $S \times S$: equal and distinct flags. Their indicator functions 1 and T form a basis, and

$$T * T = q\,1 + (q-1)T.$$

On the diagonal there are q possible intermediate lines; off the diagonal there are $q-1$.

The Hecke category

To categorify invariant kernels, use G-equivariant sheaves on pairs of flags:

$$\mathcal{H} = \mathsf{Sh}_{\mathsf{G}}(X \times X) \simeq \mathsf{Sh}(\mathsf{B}\backslash\mathsf{G}/\mathsf{B}).$$

Here Sh denotes a $\overline{\mathbb{Q}}_\ell$-linear sheaf category. On triples of flags, the projections p_{ij} define convolution

$$A \star B = R(p_{13})_!\left(p_{12}^* A \otimes p_{23}^* B\right).$$

Pull back to triples, tensor, and take cohomology over the intermediate flag. The diagonal kernel is the unit.

Thus $\mathcal{H}$ is a *monoidal category*: its product is associative and unital up to coherent equivalences. It behaves like an associative algebra among $\overline{\mathbb{Q}}_\ell$-linear categories.

Abstract trace and categorical version

A linear map $f : A \to V$ from an associative algebra satisfying $f(ab) = f(ba)$ factors through

$$A/[A,A], \qquad [A,A] = \mathsf{span}\{ab - ba\}.$$

This quotient is the *cocenter* (also called its *trace*). For $A = \mathbb{C}[\Gamma]$, its dual is the space of class functions on Γ.

A categorical analogue applied to the Hecke category recovers the unipotent character-sheaf category. of flags and convolution.

Why stable ∞-categories enter

The cocenter is the zeroth Hochschild homology of an algebra:

$$A/[A, A] \simeq \mathrm{HH}_0(A) \simeq H_0\left(A \otimes^{\mathbb{L}}_{A \otimes A^{\mathrm{op}}} A\right).$$

Here A^{op} is the opposite algebra, and the derived tensor product uses the regular A-bimodule.

For the Hecke category, we want an analogous relative tensor product of categories. Stable ∞-categories retain the complexes, homotopies, and coherence needed to define it.

We use presentable $\overline{\mathbb{Q}}_\ell$-linear categories and products preserving colimits in each variable.

∞-categories in one slide

Informally, an $(\infty, 1)$-category has objects, morphisms, morphisms between morphisms, and so on. All morphisms above level 1 are invertible up to higher homotopy.

A space X gives an example: points are objects, paths are 1-morphisms, and homotopies of paths are 2-morphisms. Continuing with higher homotopies gives an ∞-groupoid, with invertibility at every level.

Frameworks developed by Joyal, Lurie, and others make the coherent constructions precise. For a monoidal category such as $\mathcal{H}$, we can form

$$\mathsf{Tr}(\mathcal{H}) := \mathcal{H} \otimes_{\mathcal{H} \otimes \mathcal{H}^{\mathrm{rev}}} \mathcal{H},$$

where $\mathcal{H}^{\mathrm{rev}}$ has the reversed monoidal product.

Character sheaves as a categorical trace

For the Hecke category, the categorical trace has a geometric realization:

$$\mathsf{Tr}(\mathcal{H}) \simeq \mathsf{CS}_{\mathrm{unip}}(\mathsf{G}) \;\subset\; \mathsf{Sh}([\mathsf{G}/\operatorname{Ad}\mathsf{G}]).$$

The category $\mathsf{CS}_{\mathrm{unip}}(\mathsf{G})$ is generated under shifts, cones, and colimits by the character functor from the correspondence

$$\mathsf{B}\backslash\mathsf{G}/\mathsf{B} \xleftarrow{q} [\mathsf{G}/\operatorname{Ad}\mathsf{B}] \xrightarrow{p} [\mathsf{G}/\operatorname{Ad}\mathsf{G}].$$

Thus convolution on flags determines the unipotent character-sheaf category.

Ben-Zvi–Nadler establish this in the complex D-module setting; related ℓ-adic formulations include work of Ho–Li.

Deligne–Lusztig theory also as a trace

Frobenius induces an endofunctor F^* of $\mathcal{H}$. The twisted trace is realized using

$$\mathsf{B}\backslash\mathsf{G}/\mathsf{B} \xleftarrow{q} [\mathsf{G}/\operatorname{Ad}_F \mathsf{B}] \xrightarrow{p} [\mathsf{G}/\operatorname{Ad}_F \mathsf{G}].$$

The categorical form of the unipotent part of Deligne–Lusztig theory is

$$\operatorname{Tr}(\mathcal{H}, F^*) \simeq D(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G))_{\mathrm{unip}} \ \subset\ D(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G)).$$

Lang's theorem identifies $[\mathsf{G}/\operatorname{Ad}_F \mathsf{G}] \simeq [\mathrm{pt}/G]$, so its sheaf category is $D(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G))$.

Here $D(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G))$ denotes the stable category of complexes of $\overline{\mathbb{Q}}_\ell$-representations of the finite group $G = \mathsf{G}^F$.

The construction we will carry to loop groups

Finite-dimensional input	What it gives us
Flags and their cohomology	Representations
Families over $\mathbf{G}$	Sheaves and their trace functions
Convolution on pairs of flags	The Hecke category $\mathcal{H}$
The categorical trace $\mathrm{Tr}(\mathcal{H})$	Unipotent character sheaves

Next: replace k by $k((t))$ and flags by lattice flags. Can we still construct the geometry and the sheaf operations?

Some ∞-dimensional geometry

Loop groups: replace scalars by Laurent series

Let $L = k((t))$. In equal characteristic, the loop group is the functor

$$LG(R) = G(R((t))).$$

For $G = GL_2$, a point is an invertible matrix of Laurent series. The analogue of a flag is a periodic lattice chain

$$t\Lambda_0 \subset \Lambda_1 \subset \Lambda_0 \subset L^2, \qquad \dim_k(\Lambda_0/\Lambda_1) = 1.$$

Its stabilizer is an Iwahori subgroup $I \subset LG$.

The affine flag variety LG/I and the affine Hecke category $\mathcal{H}_{\mathrm{aff}} = \mathsf{Sh}(I\backslash LG/I)$ replace their finite-dimensional counterparts.

Why the geometry requires more care

Already the additive loop space has two kinds of infinity:

$$k[[t]] = \varprojlim_n k[t]/t^n, \qquad k((t)) = \varinjlim_m t^{-m} k[[t]].$$

There are infinitely many coefficients, and no uniform bound on pole order.

Loop groups can be presented by ind-schemes, but not by finite-type pieces alone. To form $[LG/\operatorname{Ad} LG]$ and its sheaves we need descent, controlled pull-push functors, and base change in this larger setting.

We first establish the sheaf operations used by the trace formalism. In mixed characteristic, the analogous construction uses Witt-vector loop groups.

p-adic representations and characters

Our result: an affine trace realized geometrically

In joint work with Xuhua He and Xinwen Zhu, we construct a fully faithful geometric realization

$$\mathsf{CS}_{\mathrm{unip}}(L\mathsf{G}) \simeq \mathsf{Tr}(\mathcal{H}_{\mathrm{aff}}) \longrightarrow \mathsf{Sh}([L\mathsf{G}/\operatorname{Ad} L\mathsf{G}]).$$

Its image is the category of unipotent affine character sheaves.

A substantial part of the work is constructing the sheaf operations on $[L\mathsf{G}/\operatorname{Ad} L\mathsf{G}]$. There are several sheaf theories in this setting, and comparing them is itself a difficult problem.

Newton strata

The affine case introduces new phenomena.

For $g \in \mathrm{GL}_n(L)$, consider its eigenvalues in an algebraic closure of L, with the valuation normalized by $v(t) = 1$. For instance:

$$g_c = \begin{pmatrix} 0 & -t \\ 1 & c \end{pmatrix}, \qquad \det(z - g_c) = z^2 - cz + t, \qquad c \in k.$$

The eigenvalue valuations are

$$\begin{array}{c|c} c \neq 0 & (0, 1) \\ c = 0 & (\frac{1}{2}, \frac{1}{2}) \end{array}$$

while $v(\det g_c) = 1$ stays constant.

Newton strata and the categories they organize

For a Newton label $\mathcal{O}$, we construct a locally closed substack

$$[L\mathsf{G}_{\mathcal{O}} / \operatorname{Ad} L\mathsf{G}] \hookrightarrow [L\mathsf{G} / \operatorname{Ad} L\mathsf{G}].$$

The underlying $L\mathsf{G}_{\mathcal{O}}$ is a locally closed ind-subscheme of the reduced loop group. Character sheaves are glued from categories on identity-containing strata in Levi loop groups, with modified twists.

Let σ act by the q-power map on coefficients and fix t. Frobenius-twisted conjugation has analogous strata

$$[L\mathsf{G}_b / \operatorname{Ad}_\sigma L\mathsf{G}] \hookrightarrow [L\mathsf{G} / \operatorname{Ad}_\sigma L\mathsf{G}], \qquad b \in B(\mathsf{G}),$$

where $B(\mathsf{G})$ indexes σ-conjugacy classes in $\mathsf{G}(k((t)))$. Their sheaf categories relate to smooth representations of the stabilizer groups $J_b(\mathbb{F}_q((t)))$; each J_b is an inner form of a Levi subgroup.

Other related work at SIMIS

Interfaces with work at SIMIS

Emile Bouaziz–Adeel Khan: elliptic loop spaces. A category with convolution serves as the input for constructing a new space encoding equivariant elliptic Hodge cohomology.

Chris Brav and collaborators: derived symplectic geometry. Calabi–Yau categories give shifted symplectic structures on moduli; local critical-locus descriptions connect to Donaldson–Thomas theory.

In each story, categorical structure makes a geometric construction possible. The QR codes lead to the research papers.

Elliptic loops (PDF)

Calabi–Yau categories (PDF)

Takeaway

Representation theory starts with questions we can ask immediately: what are the irreducible representations, and what are their characters?

For S_3 we answered by hand. For $\mathrm{GL}_2(\mathbb{F}_q)$, counting invariant lines led to cohomology, sheaves, and categorical traces.

For local-field groups such as $\mathrm{GL}_2(\mathbb{Q}_p)$, the same questions lead to infinite-dimensional geometry. My joint work with Xuhua He and Xinwen Zhu constructs affine character sheaves and organizes them using Newton strata.

Simple questions can lead us to new geometry and new categories.

Reading after the talk

Start with the example that interested you.

- Characters and S_3: Etingof et al., introductory notes.
- Geometry and topology: varieties, sheaves, and cohomology.
- Categories: Leinster, then Riehl and Antolín Camarena.
- Groups over local fields: Corinne Blondel's notes.

One-page reading route (PDF): selected starting points and small questions.

Further references (PDF)

One-page reading guide

Muito obrigado pela atenção!

Interested in doing mathematics with us?

Talk to me about PhD opportunities at SIMIS and other institutes,

or about a collaborative visit through CLACM.

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