Representability of the Derived Quot Scheme

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1 Introduction

In this short note we present some details and references for the well-known fact that the moduli of (almost) perfect complexes on a proper derived algebraic space is locally geometric. This generalizes some previous constructions as:

1. the moduli of (universally glueable) complexes on an algebraic space of Lieblich [1];

2. the derived quot scheme of Ciocan-Fontanine and Kapranov [2];

3. the the moduli of perfect complexes on a derived scheme of Töen-Vezzosi [3].

The reason why we do this, as it is clearly not an original result, is more as a study in derived algebraic geometry, more specifically on how to use Lurie's generalization of Artin's representability theorem. We should mention that the recent [4] has some overlap with the result below. The reader can also take a look at the related [5, 6, 7, 8].

1.1 Notation

- **Spc** denotes the ∞-category of spaces (we will also use this as a model for ∞-groupoids, i.e. $(\infty, 1)$-categories all of whose 1-morphisms are invertible). An element $X \in \mathbf{Spc}$ is said to be *n-truncated* if $\pi_k(X) = 0$ for all $k > n$.

- By ∞-category we mean an $(\infty, 1)$-category. Given an ∞-category $\mathscr{C}$ by the underlying ∞-groupoid $\mathscr{C}^{\sim}$ we mean the maximal ∞-subcategory which is an ∞-groupoid. Otherwise specified the categorical objects as functors, subcategories, undercategories and so on are taken in the ∞-categorical sense.

- A *derived ring* is a differential graded ring concentrated in positive cohomological degrees, similarly for R-algebras, for a fixed derived ring R. A *discrete* (or ordinary) ring (R-algebra) means a differential graded object concentrated in degree 0. We denote the category of derived rings by **Ring**.

- $\mathbf{Sch}^{\mathrm{aff}}$ denotes the ∞-category of affine derived schemes, ${}^{c\ell}\mathbf{Sch}^{\mathrm{aff}}$ denotes the subcategory whose objects are spectra of discrete commutative rings and for any $n \geq 0$ let

$\mathbf{Sch}^{\mathrm{aff},\leq n}$ be the category of affine derived schemes S given by $S = \mathrm{Spec}(R)$, where R is n-truncated as a space[1].

- Following [9] a derived scheme will be a pair $(X, \mathscr{O}_X)$ where X is an ∞-topos and $\mathscr{O}_X : \mathbf{Ring} \to X$ is a geometric morphism, such that there exists a cover $\{U_i \to \mathbb{1}_X\}$ such that each $\left(X_{/U_i}, \mathscr{O}_X\big|_{U_i}\right)$ is isomorphic to $\mathrm{Spec}(A)$ for $A \in \mathbf{Ring}$. This has a map to the category of prestacks roughly given by mapping any $A \in \mathbf{Ring}$ to $\mathrm{Maps}\left((\mathrm{Spec}(A), A), (X, \mathscr{O}_X)\right)$[2]. This map is fully faithful, however the theory is not well-behaved for arbitray ∞-topoi X. So we restrict this functor to the subcategory where X is 0-localic (cf. [10, Section 6.4.5.]). For us a *derived scheme* is an element in the image of the restricted functor. The ∞-category of derived schemes is denoted **Sch**.

- A *prestack* will mean a functor $\mathscr{X} : \mathbf{Sch} \to \mathbf{Spc}$. A *stack* is such an object satisfying étale descent. A stack $\mathscr{X}$ which admits a *smooth* surjection $U \to \mathscr{X}$ from an affine derived scheme is called an *algebraic space* (or derived Artin 0-stack). More generally, for any $n \geq 0$ a *derived Artin n-stack* is an $\mathscr{X}$ such that there exists a smooth surjective map $U \to \mathscr{X}$ from an affine derived scheme which is representable by a derived Artin $(n-1)$-stack[3].

- For a fixed derived scheme S (resp. a prestack $\mathscr{X}$) let $\mathbf{Sch}_{/S}, \mathbf{Sch}^{\mathrm{aff}}_{/S}, {}^{c\ell}\mathbf{Sch}^{\mathrm{aff}}_{/S}$ (resp. $\mathbf{Sch}_{/\mathscr{X}}, \mathbf{Sch}^{\mathrm{aff}}_{/\mathscr{X}}, {}^{c\ell}\mathbf{Sch}^{\mathrm{aff}}_{/\mathscr{X}}$) denote the corresponding subcategories inside the under category of prestacks over S (resp. $\mathscr{X}$).

- For a given prestack $\mathscr{X}$, its classical part ${}^{c\ell}\mathscr{X}$ is the Kan extension of its restriction to ${}^{c\ell}\mathbf{Sch}^{\mathrm{aff}}$. For any $n \geq 0$ we let $\tau^{\leq n}(\mathscr{X})$ denote the Kan extension of its restriction to $\mathbf{Sch}^{\mathrm{aff},\leq n}$.

- We will say a prestack $\mathscr{X}$ has a *cotangent complex* if it admits a procotangent complex in the sense of [11, Part. III.1]. Given a point $x : S \to \mathscr{X}$, where S is an affine derived scheme, we denote the cotangent complex at x by $T^*_x\mathscr{X}$. Recall this has the following corepresenting property, for any $\mathscr{F} \in \mathrm{QCoh}(S)^{\leq 0}$

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(T^*_x\mathscr{X}, \mathscr{F}) \simeq \mathrm{Maps}_{/S}(S_{\mathscr{F}}, \mathscr{X}),$$

where $S_{\mathscr{F}}$ denotes the *square-zero extension* of S associated to $\mathscr{F}$, namely $S_{\mathscr{F}} = \mathrm{Spec}(R \oplus \mathscr{F})$, for $S = \mathrm{Spec}(R)$ and $R \oplus \mathscr{F}$ the derived ring where R acts on $\mathscr{F}$ by the module structure and two elements of $\mathscr{F}$ multiply to zero.

- A prestack $\mathscr{X}$ is said to be *integrable* if for A a discrete $\mathfrak{m}$-adically complete local ring, the natural map $\mathrm{colim}_{\mathbb{N}}\mathscr{X}(A/\mathfrak{m}^n) \to \mathscr{X}(A)$ is an equivalence.

[1] Note that ${}^{c\ell}\mathbf{Sch}^{\mathrm{aff}} = \mathbf{Sch}^{\mathrm{aff},\leq 0}$.

[2] See [9, Section 2.4.] for the precise meaning of this.

[3] A map $f : \mathscr{X} \to \mathscr{Y}$ is said to be representable by an affine derived scheme (resp. algebraic space, derived Artin $(n-1)$-stack) if for all étale maps $S \to \mathscr{Y}$, $\mathscr{X} \times_{\mathscr{Y}} S$ is an affined derived scheme (resp. algebraic space, derived Artin $(n-1)$-stack).

- A map of prestacks $f : \mathscr{X} \to \mathscr{Y}$ is *locally almost of finite presentation* if locally for the étale topology it is almost of finite presentation[4].

- A map of prestacks $f : \mathscr{X} \to \mathscr{Y}$ which is representable by a derived algebraic space is said to be (strongly) *proper* if it is: (i) (strongly) separated; (ii) quasi-compact; (iii) locally of finite presentation to order 0; and (iv) For every $\mathrm{Spec}(R) \to \mathscr{Y}$ the pullback map $\mathrm{Spec}(R) \times_{\mathscr{Y}} \mathscr{X} \to \mathrm{Spec}(R)$ induces a closed map at the level of the underlying topological spaces [5].

- For a stable ∞-category $\mathscr{C}$ with a t-structure $(\mathscr{C}^{\leq 0}, \mathscr{C}^{\geq 0})$ which is left t-complete, we say an object $X \in \mathscr{C}$ is *almost compact* if: (i) it is bounded below; and (ii) for all $n \geq 0$, $\tau^{\leq n}(X)$ is a compact object in $\mathscr{C}^{\leq n}$[6].

- For a symmetric monoidal ∞-category $\mathscr{C}$ with a t-structure, we denote by $\mathscr{C}^0$ its heart, or more generally for $I \subset \mathbb{Z}$ let $\mathscr{C}^I$ be the subcategory generated by objects which have vanishing cohomology for any $i \notin I$. For $n \geq 0$ we say that an element $\mathscr{F} \in \mathscr{C}$ has *Tor amplitude bounded by* n if for all $\mathscr{G} \in \mathscr{C}^0$

$$\pi_k (\mathscr{F} \otimes \mathscr{G}) = 0,$$

  for all $k \geq n$. In particular, for $n = 0$ we say $\mathscr{F}$ is *flat*. And if there exists an n such that $\mathscr{F}$ has Tor amplitude bounded by n, we just say that $\mathscr{F}$ has *finite Tor amplitude*.

- Given a map $f : X \to S$ or an element $\mathscr{F} \in \mathrm{QCoh}(X)$, and a map $T \to S$, we denote by $X_T = T \times_S X$ its base change and $\mathscr{F}_T$ the pullback of $\mathscr{F}$ to X_T.

2 Representability theorem

Let $S \in \mathbf{Sch}^{\mathrm{aff}}$ be the spectrum of a connective derived ring R. We say that R is a *Grothendieck ring* if it is Noetherian as a derived ring[7] and for all prime ideals $\mathfrak{p} \subset H^0(R)$, the map from the localization of $H^0(R)$ at $\mathfrak{p}$ to its completion is geometrically regular. Consider $\mathscr{X} \to S$ a prestack over S, one has the following theorem due to Lurie [14].

Theorem 1. *$\mathscr{X}$ is an n-derived Artin stack if and only if*

(i) $\mathscr{X}$ has a cotangent complex;

(ii) $\mathscr{X}$ is infinitesimally cohesive;

[4]For a map $A \to B$ of derived rings, we say B is almost of finite presentation over A if for all $n \geq 0$ it is of finite presentation to order n. Roughly, this means that $\tau^{\leq n}(B)$ is a compact object in the category of $\tau^n(A)$-algebras (for a precise definition cf. [12, Section 8.]).

[5]See [13, Section 1.4.] for more on the notion of underlying points of a derived algebraic space.

[6]Here $\tau^{\leq n}$ denotes the truncation of X to the subcategory $\mathscr{C}^{\leq n}$. Note this is coherent with the previous notaiton for the usual t-structure on the category of connective derived rings. Recall an object is compact if the functor it corepresents commutes with filtered colimits.

[7]Recall this means that $H^0(R)$ is Noetherian and $H^i(R)$ is finitely generated as a $H^0(R)$-module for each $i > 0$.

(iii) $\mathscr{X}$ *is convergent;*

(iv) $\mathscr{X}$ *satisfies étale descent;*

(v) $\mathscr{X}$ *is locally almost of finite presentation;*

(vi) $\mathscr{X}(T)$ *is n-truncated, for any* $T \in^{c\ell} \mathbf{Sch}^{aff}_{/S}$*;*

(vii) $\mathscr{X}$ *is integrable.*

One can use some results of [11] to reduce most of these conditions to properties of ${}^{c\ell}\mathscr{X}$ and of the deformation theory of $\mathscr{X}$.

Theorem 2. $\mathscr{X}$ *is an n-derived Artin stack if and only if*

(1) $\mathscr{X}$ *admits deformation theory;*

(2) $T^*_x\mathscr{X}$ *has finitely generated cohomology on each degree, for all* $x : T \to \mathscr{X}$*, where* $T \in \mathbf{Sch}^{aff}_{/\mathscr{X}}$*;*

(3) ${}^{c\ell}\mathscr{X}$ *satisfies étale descent;*

(4) ${}^{c\ell}\mathscr{X}$ *is locally almost of finite presentation;*

(5) $\mathscr{X}(T_0)$ *is n-truncated, for any* $T_0 \in^{c\ell} \mathbf{Sch}^{aff}_{/S}$*;*

(6) $\mathscr{X}$ *is integrable.*

Proof. Item (1) is by definition conditions (ii-iii) and a weaker form of condition (i) (cf. [11, Part III.1 - Definition 7.1.2]). However, condition (2) implies that the a priori pro-cotangent complex of $\mathscr{X}$ is corepresentable. Thus, one establishes that (i-iii) is equivalent to (1-2).

Items (1-3) are equivalent to (i-iv). Indeed, this is a result from [11](cf. Chapter III, Section 1, Proposition 8.2.2 and Remark 8.2.4) which says that if a prestack admits deformation theory and its classical part satisfies étale descent, then so does it.

Items (1-2) and (4) imply (v). This is [11, Part III.1 - Corollary 9.1.4.]. Conversely, (v) clearly implies (4).

The last two items just match the previous conditions. □

Remark 1. Conditions (1-2) is what [11] refers to as $\mathscr{X}$ admits corepresentable deformation theory.

3 The Derived Quot functor of points

Let R be an excellent derived ring, and S the corresponding affine derived scheme. Let $X \to S$ be a (strongly) proper map of finite type representable by a derived algebraic space. We denote by $\mathrm{QCoh}(X)^{\mathrm{aperf}}$ the ∞-subcategory of $\mathrm{QCoh}(X)$ consisting of almost perfect objects, that is $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{aperf}}$ if $\mathscr{F}$ is bounded above and for all $n \in \mathbb{Z}$, $\tau^{\geq n}(\mathscr{F})$ is a compact object of $\mathrm{QCoh}(X)^{\geq n}$.

Remark 2. The category $\mathrm{QCoh}(X)^{\mathrm{aperf}}$ is the good object to consider because almost perfect objects are preserved under proper maps. In contrast, the subcategory $\mathrm{Coh}(X)$ of complexes with bounded and coherent cohomology does not behave well in derived algebraic geometry. Namely, if one considers the pullback of the structure sheaf of any affine derived scheme to its dual numbers, this has only unbounded below resolutions, which are neither coherent nor perfect; they are, however, almost perfect.

We also record here some properties about $\mathrm{QCoh}(X)^{\mathrm{aperf}}$ that we will use at times (cf. [13] Proposition 5.2.4.).

Proposition 1. *The category $QCoh(X)^{aperf}$ has a t-structure, given by*

$$QCoh(X)^{\mathrm{aperf},\geq \mathrm{n}} = QCoh(X)^{\geq n} \cap QCoh(X)^{aperf}.$$

In particular, $QCoh(X)^{aperf,0} \simeq Coh(X)^0$, i.e. the heart is the abelian category of coherent sheaves. An element $\mathscr{F} \in QCoh(X)^{aperf}$ is a perfect complex if and only if $\mathscr{F}$ has finite Tor amplitude.

From now on, we will use $\left(\mathrm{QCoh}(X)^{\mathrm{aperf},\leq 0}, \mathrm{QCoh}(X)^{\mathrm{aperf},\geq 0}\right)$ to refer to the above t-structure.

Fix $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{aperf},\geq 0}$, one can study the following functors $\mathscr{X}_i^{(n)} : \mathbf{Sch}^{\mathrm{aff}}_{/S} \to \mathbf{Grpd}$ for $i = 1, 2$ and $n \in \mathbb{N} \cup \{\infty\}$.

1. $\mathscr{X}_1^{(n)}(T)$ is the underlying ∞-groupoid associated to the ∞-subcategory of $\mathrm{QCoh}(X_T)^{\mathrm{aperf}}$ consisting of sheaves $\mathscr{F}$, such that for all étale covers $T' \to T$

$$\Gamma\left(X_{T'}, \mathscr{H}\mathrm{om}_{X_{T'}}\left(\mathscr{F}_{T'}, \mathscr{F}_{T'}\right)\right)[-1]$$

   has Tor amplitude bounded by n.

2. $\mathscr{X}_2^{(n)}(T)$ is the underlying ∞-groupoid associated to the ∞-subcategory of the undercategory $\mathrm{QCoh}(X_T)^{\mathrm{aperf}}_{\mathscr{F}_T/}$ consisting of objects $\mathscr{F}_T \overset{q}{\twoheadrightarrow} \mathscr{G}$, such that for all étale covers

$$\Gamma\left(X_{T'}, \mathscr{H}\mathrm{om}_{X_{T'}}\left(\mathscr{G}_{T'} \otimes \ker(q)_{T'}\right)\right)$$

   has Tor amplitude bounded by n.

Remark 3. If $n = \infty$ then the boundedness condition on the Tor amplitude is vacuous so one retains the ∞-groupoid associated to the whole category. If one suppose that we take elements $\mathscr{F}$ which are perfect, then the conditions can be written as $\mathscr{F} \in \mathrm{QCoh}(X)^{\mathrm{aperf}}_{\leq n}$ if for all étale covers $T' \to T$

$$\Gamma\left(X_{T'}, \mathscr{H}\mathrm{om}_{X_{T'}} \mathscr{F}^{\vee}_{T'} \otimes \mathscr{F}_{T'}\right)[-1],$$

as the dual $\mathscr{F}^{\vee}$ is well-defined. For $n = 1$ the objects of this category are called *universally glueable* by [1]. The above is not such a hard condition to be satisfied. For instance if $X \to S$ is a flat, proper, perfect and classical algebraic space over S it is automatic from [4, 15]. We refer the reader to [4] for more general examples.

We can now state the main results of this note.

Theorem 3. *For $i = 1, 2$, the functors $\mathscr{X}_i^{(n)}$ are derived Artin n-stacks locally of finite type, and $\mathscr{X}_i^{(\infty)}$ are locally geometric stacks*[8]*. In particular*

$$\mathscr{X}_i^{(\infty)} \simeq colim_{\mathbb{N}} \mathscr{X}_i^{(n)}.$$

To put the above result in context here are some comparisons with previous constructions in the literature. Let $\mathscr{X}_3$ be the substack of $\mathscr{X}_2^{(0)}$ associated to the subcategory of perfect complexes concentrated in non-negative degree.

Theorem 4. *When $X \to S$ is a projective ordinary scheme. The derived Artin 0-stack $\mathscr{X}_3$ agrees with the derived scheme associated to the dg scheme constructed by Ciocan-Fontanine and Kapranov. In particular, it is a derived scheme.*

Theorem 5. *When $X \to S$ is an ordinary algebraic space. The underlying classical stacks of $\mathscr{X}_1^{(1)}$ coincides with the moduli of complex of universally glueable S-perfect sheaves of Lieblich[1].*

Our strategy of proof is rather straightforward, we just check that the conditions of Theorem 2. are satisfied. To give a better understanding of the structure of the result we will start by proving the conditions that hold regardless of n first. We do that for the particular case of $\mathscr{X}_1^{(\infty)}$ and $\mathscr{X}^{(\infty)_2}$, which turn out to be rather general as all the other $\mathscr{X}_1^{(n)}$ and $\mathscr{X}_2^{(n)}$ turn out to be open substacks of these, so that they automatically satisfy the conditions. Finally we check the truncatedness condition for each class.

Actually, we will prove some more general results above the assignment at the level of ∞-categories which we then by abstract nonsense imply the result for the functors $\mathscr{X}_1^{(\infty)}$ and $\mathscr{X}_2^{(n)}$. More precisely, consider

$$\mathscr{A}\mathrm{Perf}(X) : \mathbf{Sch}_{/S}^{\mathrm{aff}} \to \mathbf{Cat}_\infty,$$

which assigns $\mathscr{A}\mathrm{Perf}(X)(T) = \mathrm{QCoh}(X_T)^{\mathrm{aperf}}$ to objects and pullbacks for morphisms.

Proposition 2. *The functor $\mathscr{A}$Perf has corepresentable deformation theory.*

Proof. The first thing to check is that

$$\mathrm{colim}_{\mathbb{N}} \mathrm{QCoh}^{\mathrm{aperf}}(X_{\tau^{\leq n}(T)}) \xrightarrow{\simeq} \mathrm{QCoh}^{\mathrm{aperf}}(X_T),$$

where $\tau^{\leq n}(T)$ means the derived scheme associated to the truncation of the derived ring to cohomological degrees less than or equal to n. It suffices to check that given $\mathscr{F} \in \mathrm{Coh}(X_T)$ there exists an $m \geq 0$ and a $\mathscr{G}_m \in \mathrm{QCoh}^{\mathrm{aperf}}(X_{\tau^{\leq m}(T)})$ such that

$$\mathscr{F} \simeq q_m^* \mathscr{G}_m,$$

[8] This is called D^- locally geometric in [3]. It roughly means that locally one can find a derived Artin n-stack which is isomorphic to it. And that globally one can find a filtered sequence of derived Artin stacks whose colimit is the corresponding functor.

where $q_m : \tau^{\leq m}(T) \to T$. It is enough to argue locally, so one can suppose that X_T is affine. Now we argue by induction on the length of $\mathscr{F}$. For $\mathscr{F}$ of length 0, we know that the almost perfect condition implies that $\mathscr{F}$ is a finitely presented module (cf. [16]). Now since $\mathscr{F}$ is finitely presented, there exists a number m such that all the relations between the finite generators of $\mathscr{F}$ can be written by considering only elements of cohomology degree less than or equal to m in T. One takes $\mathscr{G}_m$ to be the module over $\tau^{\leq m}(T)$ generated by these generators and relations. It is clear that it's base change to T agrees with $\mathscr{F}$. A similar argument proves that morphisms between objects are determined at a finite stage as well. Now suppose we proved the result, for both objects and morphisms, for up to length n. As we can form any almost perfect complex $\mathscr{F}$ of length $n+1$ by an extension of a complex of length n by something of length 1 we just need that the extensions are also determined at a finite level. This follows from the induction as well.

Secondly, we need to check that it admits a pro-cotangent complex, i.e. it takes some special pushouts to pullbacks, namely

$$\mathrm{QCoh}^{\mathrm{aperf}}(X_{S \sqcup_{S_{\mathscr{G}_1}} S_{\mathscr{G}_2}}) \xrightarrow{\simeq} \mathrm{QCoh}^{\mathrm{aperf}}(X_S) \times_{\mathrm{QCoh}^{\mathrm{aperf}}(X_{S_{\mathscr{G}_1}})} \mathrm{QCoh}^{\mathrm{aperf}}(X_{S_{\mathscr{G}_2}}),$$

where $\mathscr{G}_1 \to \mathscr{G}_2$ is a map in $\mathrm{QCoh}(S)^{\leq 0}$, which is surjective in H^0. We construct an inverse to the natural map above, that is given $\mathscr{K} \in \mathrm{QCoh}^{\mathrm{aperf}}(X_S)$, $\mathscr{K}_1 \in \mathrm{QCoh}^{\mathrm{aperf}}(X_{S_{\mathscr{G}_1}})$ and $\mathscr{K}_2 \in \mathrm{QCoh}^{\mathrm{aperf}}(X_{S_{\mathscr{G}_2}})$ whose pullback to X_S agree with $\mathscr{K}$ we consider the kernel of

$$\imath_{S_{\mathscr{G}_1},*}\mathscr{K}_1 \oplus \imath_{S_{\mathscr{G}_2},*}\mathscr{K}_2 \to \imath_{S,*}\mathscr{K},$$

where $\imath_S : S \to S_{\mathscr{G}}, \imath_{S_{\mathscr{G}_1}}, \ldots$ are the natural inclusions. It is easy to check that this is actually an inverse.

The last thing to check is infinitesimal cohesiveness, i.e. for any $\mathscr{G} \in \mathrm{QCoh}(S)^{\leq -1}_{T^*(S)/}$ the following holds

$$\mathrm{QCoh}^{\mathrm{aperf}}(X_{S'}) \xrightarrow{\simeq} \mathrm{QCoh}^{\mathrm{aperf}}(X_S) \times_{\mathrm{QCoh}^{\mathrm{aperf}}(X_{S_{\mathscr{G}}})} \mathrm{QCoh}^{\mathrm{aperf}}(X_S),$$

where $S' \equiv S \sqcup_{S_{\mathscr{G}}} S$ is the (non-split) square zero extension corresponding to $\mathscr{G}$. The same map as above gives an inverse construction. ☐

Proposition 3. *The functor $\mathscr{A}$Perf satisfies étale descent.*

Proof. One can realise $\mathscr{A}\mathrm{Perf}(T)$ as a $\mathrm{QCoh}(T)$-linear category in the symmetric monoidal ∞-category of stable ∞-categories. Then by [17, Theorem. 5.4.] it satisfies étale descent. ☐

The above actually uses a lot of machinery for what we want to do. We can prove more directly this results if we restrict to our functors $\mathscr{X}_i^{(\infty)}$, for $i = 1, 2$.

Proposition 4. *The functors ${}^{cl}\mathscr{X}_1^{(\infty)}$ satisfies étale descent, more generally $\mathscr{X}_i^{(\infty)}$ itself satisfies étale descent, for $i = 1, 2$.*

Proof. We note that ${}^{c\ell}\mathscr{X}_1^{(\infty)}(T)$ is equivalent to $\mathrm{QCoh}^{\mathrm{aperf}}(\tau^{\leq 0}(X_T))$, so one has to check that the ∞-category of almost perfect complexes on an ordinary scheme satisfy étale descent. This is proved for example in [18, Theorem 1.3.4.][9]. Or more explicitly still one can recall how the proof of the usual descent for the ordinary (abelian) category of coherent sheaves on X goes, i.e. by a Bar-Beck argument with respect to the descent data category, and run the same argument for ∞-categories using the analogous Bar-Beck-Lurie theorem. Now we quote a result of [11] (Part III.1 - Proposition. 8.2.2. and Remark. 8.2.4.), which says that it is enough to check étale descent for discrete affine schemes, so we get the result for $\mathscr{X}_1^{(\infty)}$. The case of $\mathscr{X}_2^{(\infty)}$ is completely analogous. □

Proposition 5. *The functor $\mathscr{A}Perf$ is integrable*[10].

Proof. This is an extension of Grothendieck's existence result to derived algebraic geometry. This is the main result of [13] (cf. Theorem 5.3.2), namely one has an equivalence of categories

$$\mathrm{QCoh}(X_{\hat{A}})^{\mathrm{aperf}} \overset{\imath^*}{\simeq} \mathrm{QCoh}(X_A)^{\mathrm{aperf}},$$

where $X_{\hat{A}}$ is the formal completion of X_A along $X_{A/m}$, and $\imath : X_{\hat{A}} \to X_A$ is the induced inclusion map. The equivalence preserve perfect complexes, since it preserves perfect objects and by previous results of [13] it also respects the t-structures. □

By restricting the equivalence to the associated ∞-groupoids one has the following.

Corollary 1. *The functors $\mathscr{X}_i^{(\infty)}$ for $i = 1, 2$ are integrable as well.*

Remark 4. There is a more direct proof of the above result based on the usual Grothendieck existence theorem. We give it below for convinience of the reader not so familiar with the derived result. Let A be an complete discrete local ring with maximal ideal $\mathfrak{m}$. We denote by $X_{\hat{A}}$ the base change of X to the completion of $\mathrm{Spec}(A)$ at the point $\mathfrak{m}$. This has a natural inclusion $\imath : X_{\hat{A}} \to X_A$, which induces a restriction functor $\imath^* : \mathrm{QCoh}^{\mathrm{aperf}}(X_A) \to \mathrm{QCoh}^{\mathrm{aperf}}(X_{\hat{A}})$. We will prove this is an equivalence by constructing an inverse. By Proposition. 1. we know that $\mathrm{QCoh}^{\mathrm{aperf}}(X_A)$ has a t-structure $\left(\mathrm{QCoh}^{\mathrm{aperf},\geq 0}(X_A), \mathrm{QCoh}^{\mathrm{aperf},\leq 0}(X_A)\right)$. Since the functor $\imath^*$ is t-exact, it is enough to prove that its restriction to $\mathrm{QCoh}^{\mathrm{aperf},\geq 0}(X_A)$ is an equivalence. Let

$$\mathrm{QCoh}^{\mathrm{aperf},[0,n]}(X_A) = \mathrm{QCoh}^{\mathrm{aperf},\geq 0} \cap \mathrm{QCoh}^{\mathrm{aperf},\leq n}(X_A),$$

we will proceed by induction on n. For $n = 0$, we have that $\mathrm{QCoh}^{\mathrm{aperf},0}(X_A)$ is the ordinary abelian category of coherent sheaves on X_A. Now for any derived algebraic space, this is

[9] This reference actually proves a stronger result, namely that $\mathrm{QCoh}(X)$ satisfy fppf descent as a sheaf of DG-categories. To get our version one just restrict to the underlying ∞-groupoid, consider almost perfect objects, and restrict to covers which are faithfully flat and étale.

[10] This means that when $\mathscr{X}_1$ is restricted to the category ${}^{c\ell}\mathbf{Sch}_{/S}^{\mathrm{aff,formal}}$ of Artinian discrete derived rings one has

$$\mathscr{X}_1(\mathrm{Spec}(A)) \simeq \lim_{\mathbb{N}} \mathscr{X}_1(\mathrm{Spec}(A/m^n)),$$

for A such a derived ring with maximal ideal m.

equivalent to the usual category of coherent sheaves on the underlying underived algebraic space (cf. (?)). Thus the equivalence

$$\mathrm{QCoh}^{\mathrm{aperf},0}(X_A) \simeq \mathrm{QCoh}^{\mathrm{aperf},0}(X_{\hat{A}})$$

follows from the usual Grothendieck existence theorem (cf. []). Suppose that the result holds for $\mathrm{QCoh}^{\mathrm{aperf},[0,m]}(X_A)$, and let $\imath_m^*$ be the restriction of $\imath^*$ to such a category. Then given any $\mathscr{F} \in \mathrm{QCoh}^{\mathrm{aperf},[0,m+1]}(X_{\hat{A}})$ we will construct an element $\Psi(\mathscr{F}) \in \mathrm{QCoh}^{\mathrm{aperf},[0,m+1]}(X_A)$ by

$$\mathrm{Coker}\left((\imath_m^*)^{-1}(\mathscr{K}[1]) \to (\imath_m^*)^{-1}(\tau_{\leq m}(\mathscr{F}))\right),$$

where $\tau_{\leq m}(\mathscr{F})$ is the truncation of $\mathscr{F}$ to degrees less than or equal to m and $\mathscr{K}$ is the cokernel of the natural inclusion $\tau_{\leq m}(\mathscr{F}) \to \mathscr{F}$. One checks easily that Ψ is an inverse to $\imath_{m+1}^*$. This finishes the proof.

Lemma 1. *Given any map $x : T \to \mathscr{X}_1^{(\infty)}$, that is an element $\mathscr{F}_T \in QCoh^{aperf}(X_T)$, the cotangent space at x is*

$$T_x^*\mathscr{X}_1^{(\infty)} \simeq Hom_{X_T}(\mathscr{F}_T, \mathscr{F}_T[1]).$$

Given any map $x : T \to \mathscr{X}_2^{(\infty)}$, that is a map $\mathscr{F}_T \xrightarrow{q} \mathscr{G}_T$, the cotangent space at x is

$$T_x^*\mathscr{X}_2^{(\infty)} \simeq Hom_{X_T}(\mathscr{G}_T, Ker(q)_T[1]).$$

Proof. Since $\mathrm{QCoh}^{\mathrm{aperf}}(X_T)$ satisfies descent this is a local question, by picking an étale cover of X one can restrict to the affine case, i.e. $X = \mathrm{Spec}(A)$ for some R-algebra A. Now the result follows from Corollary 3. Let's consider the map $\mathscr{X}_2^{(\infty)} \xrightarrow{f} \mathscr{X}_1^{(\infty)} \times \mathscr{X}_1^{(\infty)}$ which forgets the map between the complexes. For any $x : T \to \mathscr{X}_2^{(\infty)}$, this induces a map

$$T_{f(x)}^*\left(\mathscr{X}_1^{(\infty)} \times \mathscr{X}_1^{(\infty)}\right) \to T_x^*\mathscr{X}_2^{(\infty)},$$

which is surjective. Given $\psi_2 : \mathscr{G}_T \to \mathscr{G}_T[1]$, it is an element of the cotangent space of $\mathscr{X}_2^{(\infty)}$ if there exists and element $\psi_1 : \mathscr{F}_T \to \mathscr{F}_T[1]$ such that the following commutes By

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\psi_1} & \mathscr{F}[1] \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q[1]} \\ \mathscr{G} & \xrightarrow[\psi_2]{} & \mathscr{G}_T\ [1] \end{array}$$

the surjectivity we can always find ψ_1 that makes it commutes. Fixing such an element the space of ψ_2 which fit in the diagram is a torsor for $\mathrm{Hom}_{X_T}(\mathscr{G}_T, \mathrm{Ker}(q)_T[1])$. This finishes the proof. □

Corollary 2. *The functor $\mathscr{X}_i^{(\infty)}$ satisfies conditions (2) and (4) of Theorem 2, for $i = 1, 2$.*

Proof. Condition (2) follows from the above lemma and the fact that the cohomology of an almost perfect complex of sheaves for a proper map between derived algebraic spaces is finitely generated [17, Theorem 3.2.2.][11]. Condition (4) is just a rephrasing of the definition of $\mathrm{QCoh}^{\mathrm{aperf}}(X)$. Indeed, one says that a prestack $\mathscr{X}$ if locally almost of finite presentation if and only if for all étale map $T \to S$ $\mathscr{X}(T) \to T$ is locally almost of finite presentation. This is equivalent to: for all $n \geq 0$, and for all $\{T_i\}_I$ a filtered collection in ${}^{\mathrm{aff}}\mathbf{Sch}_{/S}$

$$\tau^{\leq n}(\mathscr{X})(\mathrm{lim}_I T_i) \overset{\simeq}{\to} \mathrm{lim}_I \tau^{\leq n}(\mathscr{X})(T_i)$$

is an isomorphism. The last condition for $\mathscr{X}_i^{(\infty)}$ ($i = 1$ or 2) is exactly that the objects of $\mathrm{QCoh}^{\mathrm{aperf}}(X)$ are perfect when restricted to $\mathrm{QCoh}^{\mathrm{aperf}}(X)^{\leq n}$ for all n, which is the case by definition. □

Finally we need to check the truncatedness condition, namely condition (5) of 2. Before we make a remark.

Remark 5. One would be naive to expect that we can represent $\mathscr{X}_i^{(\infty)}$ (for $i = 1, 2$) as a derived Artin n-stack for some fixed n. Namely, condition (5) boils down to the following. For any $m \geq n + 2$, $T \in^{c\ell,\mathrm{aff}} \mathbf{Sch}_{/S}$ a diagram as can be lifted by a map $\Delta_m \to \mathscr{X}_1^{(\infty)}$.

$$\begin{array}{ccc} \partial\Delta_m & \longrightarrow & \Delta_m \\ \downarrow & & \\ \mathscr{X}^{(\infty)}{}_1 \quad (T) & & \end{array}$$

Concretely, what this says is that given $\mathscr{F} \in \mathrm{QCoh}^{\mathrm{aperf}}(X_T)$ the space

$$\mathrm{Hom}_{X_T}(\mathscr{F}_T, \mathscr{F}_T[m-1])$$

is discrete, i.e. $\pi_k(\mathrm{Hom}_{X_T}(\mathscr{F}, \mathscr{F}[m-2])) \simeq 0$ for all $k > 0$. For the case of $\mathscr{X}_2^{(\infty)}(T)$ one asks that $\mathrm{Hom}_{X_T}(\mathscr{G}_T, \mathrm{Ker}(q)_T[m])$ is discrete.

Proposition 6. *The functors $\mathscr{X}_i^{(m)}$ satisfy condition (5) with $n = m$, for all $m \geq 0$ and $i = 1, 2$.*

Proof. From the previous remark we see that the defining condition of $\mathscr{X}_1^{(m)}(T)$ ensures that

$$\pi_k(\mathrm{Hom}_{X_T}(\mathscr{F}_T, \mathscr{F}_T[m-1])) \simeq \pi_{k+m}(\mathrm{Hom}_{X_T}(\mathscr{F}_T, \mathscr{F}_T)[-1])$$

vanishes for $k + m > m$. Similarly, for $\mathscr{X}_2^{(m)}(T)$. □

[11] More precisely, the pushforward by a proper map preserves almost perfect objects, which over an affine derived scheme are finitely presented modules on each degree (cf. [16] Proposition. 7.2.5.17.).

4 Comparison with Ciocan-Fontanine and Kapranov construction

We will now prove 4. Before doing that let's make a little more explicit what the Tor amplitude condition on $\mathscr{X}_3$ implies.

Remark 6. To simplify our discussion suppose $\mathscr{F} \in \mathbf{Perf}^0(X)$, i.e. that it is a coherent sheaf (complex concentrated in degree 0) on X. Then any map $T \to \mathscr{X}_3$ determines a map $q : \mathscr{F} \to \mathscr{G}$. The Tor amplitude condition asks, in particular, that

$$\mathrm{Hom}_{X_T}(\mathscr{G}, \mathrm{Ker}(q))$$

has Tor-amplitude 0, i.e. is flat. But we also restricted to $\mathscr{G}$ that are concentrated in non-negative degrees. Both of these conditions combine to give that $\mathrm{Ker}(q)$ is concentrated in degree 0, that is $\mathscr{F} \to \mathscr{G}$ is surjective and $\mathscr{G}$ itself is flat. These are exactly the conditions in the definition of the usual Quot functor of points.

Also in [2] they consider the derived quot scheme defined over a field. To agree with their construction in this section we take our excellent ring R to coincide with their field.

Let $\mathscr{Y}$ be the prestack associated to the dg-scheme $\mathscr{D}\mathrm{Quot}_X(\mathscr{F})$ constructed in [2].

Proposition 7. *$\mathscr{Y}$ is a derived scheme and it has a tautological object $\mathscr{T} \in \mathbf{Perf}(\mathscr{Y} \times_S X)^0_{\mathscr{F}_{\mathscr{Y} \times_S X}/}$ which is also flat.*

Before giving the proof we state a more concrete discription of the derived schemes we consider.

Remark 7. Let $\mathscr{Y} = \mathbf{Shv}(Y)$ be the 0-localic ∞-topos corresponding to a derived scheme $(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}})$. Then the condition of being a derived scheme is equivalent to the following (cf. [9] Theorem. 4.2.15.)

(i) $(Y, \pi_0(\mathscr{O}_{\mathscr{Y}}))$ is an ordinary scheme;

(ii) For all $i > 0$, $\pi_i(\mathscr{O}_Y)$ is a quasi-coherent $\pi_0(\mathscr{O}_{\mathscr{Y}})$-module;

(iii) the sheaf $\mathscr{O}_{\mathscr{Y}}$ satisfies descent with respect to hypercovers.

Here one considers covers with respect to the Zariski topology.

Proof. Recall the definition of a dg-scheme as in [2] is a pair $(X, \mathscr{O}_X)$ where $(X, \pi_0(\mathscr{O}_X))$ is an ordinary scheme, and each $\pi_i(\mathscr{O}_X)$ is a quasi-coherent $\pi_0(\mathscr{O}_X)$-module. This will be an element of **Sch** if $\mathscr{O}_X$ is a hypersheaf, i.e. satisfy descent with respect to hypercovers. Now we invoke a result from [10] Section. 7.4. which says that for any Noetherian topological space of finite Krull dimension any sheaf is a hypersheaf. Then we note that by construction Y is the underlying space of the scheme representing the usual quot functor of points. Than, for $X \to S$ projective, over S Noetherian the result of Grothendieck [19] states that Y is locally of finite type over S. This implies that it is Noetherian as locally it is of finite type over a Noetherian space and also that it has finite Krull dimension[12] the locally of finite type condition gives that each affine cover $\mathrm{Spec}(A)$ has A of finite relative Krull dimension over R. However, [2] assumption that R is a field, hence has Krull dimension 0, gives that A has finite (absolute) Krull dimension. So we can apply Lurie's result and all sheaves are hypersheaves. Let $\mathscr{Y}$ be the image of the dg-scheme constructed in [2] under this inclusion. We claim $\mathscr{Y}$ has a tautological object $\tilde{\mathscr{T}} \in \mathrm{QCoh}^{\mathrm{aperf}}(\mathscr{Y})^{\mathscr{M}/}$, where $\mathscr{M}$ is the sheaf associated to the module $\oplus_{\mathbb{N}} H^0(X, \mathscr{F}(n))$ and that $\mathscr{T}$ is flat. It sufficies to determined a bounded complex of sheaves of quasi-coherent $\pi_0(\mathscr{O}_{\mathscr{Y}})$-modules on Y to determined an element of $\mathrm{QCoh}^{\mathrm{aperf}}(\mathscr{Y})$. From the construction of [2] one knows Y has a tautological quotient sheaf $\mathscr{M} \overset{q_{\mathscr{T}}}{\to} \tilde{\mathscr{T}}$, where $\tilde{\mathscr{T}}$ is the locally free sheaf whose stalk at a point $x \in \mathscr{D}\mathrm{Quot}_X(\mathscr{F})$ is the element $\mathscr{G}$ corresponding to $\mathscr{F} \overset{q}{\to} \mathscr{G}$. This clearly receives a map from the sheaf $\mathscr{M}$. Our sheaf $\mathscr{T}$ is the pullback of $\tilde{\mathscr{T}}$ to $\mathscr{Y} \times_S X$. It is obviously flat as it is a locally free sheaf. □

Remark 8. In [2] they only consider $\mathscr{F} \in \mathrm{QCoh}^{\mathrm{aperf},0}(X)$, i.e. $\mathscr{F}$ is a genuine coherent sheaf, not a complex of coherent sheaves. This is why in our previous analysis of $\mathscr{X}_3$ we supposed the same on $\mathscr{F}$. However one could have modified their construction to a general $\mathscr{F} \in \mathrm{QCoh}^{\mathrm{aperf},\geq 0}(X)$ by taking $\oplus_{i\in\mathbb{Z}} \oplus_{n\geq 0} H^0(X, \mathscr{H}^i(\mathscr{F})(n))$. Note that the Tor amplitude 0 condition gives that $\mathscr{F}$ and $\mathscr{G}$ have isomorphic cohomology except for one degree, on which the former surjects on the later.

Remark 9. The above produces a map between prestacks $f : \mathscr{Y} \to \mathscr{X}_3$. Indeed, since $\tilde{\mathscr{T}}$ by definition is S-flat and concentrated in non-negative degrees, by the universal property of $\mathscr{X}_3$ we get the map f.

Proposition 8. *The map f induces an isomorphism at the level of classical prestacks, and for every $y \in^{c\ell} \mathbf{Sch}^{aff}_{/sY}$ the induced map $T^*_{f(y)}\mathscr{X}_3 \to T^*_y\mathscr{Y}$ is an isomorphism.*

Proof. The first statement is the fact that ${}^{c\ell}\mathscr{X}_3(T)$ coincides with the usual quot functor of points, for any $T \in^{c\ell} \mathbf{Sch}^{\mathrm{aff}}_{/S}$ and $\mathscr{F} \in \mathrm{QCoh}^{\mathrm{aperf},0}(X)$. This is what we made explicit in Remark. 6. The second statement is the claim that for any $\mathscr{F} \overset{q}{\to} \mathscr{G}$ and $i \in \mathbb{Z}$ one has the following isomorphisms

$$H^i\left(T^*_y\mathscr{Y}\right) \simeq \mathrm{Hom}_R(M_{\ker(q)}, M_{\mathscr{G}}[i]) \simeq \mathrm{Hom}_{\mathscr{Y}\times_S X}(\mathrm{Ker}(q_{\mathscr{T}}), \mathscr{T}[i]),$$

where $M_{\ker(q)} = \oplus_{\mathbb{N}} H^0(X, \ker(q)(n))$ and similarly for $M_{\mathscr{G}}$. The first isomorphism is just a standard calculation, while the second isomorphism is just a restatement of [2, Proposition 4.3.3.]. □

Proof of Theorem 4. We can now invoke another nice result of [11] (cf. Part III.1 - Proposition 8.3.2.). It says that if one has a map between two prestacks with deformation theory (note that $\mathscr{Y}$ has deformation theory because it is a derived scheme locally almost of finite type), which induces an isomorphism between the classical prestacks and between the cotangent complexes at all points, then the original map was an isomorphim. This is exactly what we verified in the previous proposition. □

[12] Recall the Krull dimension of a scheme can be taken to be the supremum of the Krull dimension of affine schemes covering it.

A Tangent space calculation

Let A be a derived ring, M be an A-module concentrated in non-negative degrees. We denote by $A \oplus M$ its square-zero extension by M. This has a map $A \oplus M \to A$ of A-algebras. Given any $L \in \mathrm{Mod}_A$, the ∞-category of A-modules one can consider $\eta_L : L \otimes_{A\oplus M} A \to L$ the adjunction corresponding to the functors between Mod_A and $\mathrm{Mod}_{A\oplus M}$ which forget the structure of A-module through the quotient above, and extension of scalar with respect to the same map. Let Q be the kernel of the map η_L, i.e.

$$Q \xrightarrow{\imath} L \otimes_{A\oplus M} A \xrightarrow{\eta_L} L$$

is an exact sequence.

Given any L, one has a section of the map $\eta_{L\otimes_{A\oplus M} A} : L \otimes_{A\oplus M} A \otimes_{A\oplus M} A \to L \otimes_{A\oplus M} A$. Namely, consider

$$M \to A \oplus M \to A,$$

as an exact sequence in $\mathrm{Mod}_{A\oplus M}$. One can tensor it with $L \otimes_{A\oplus M}$ to obtain

$$L \otimes_{A\oplus M} M \to L \to L \otimes_{A\oplus M} A. \tag{1}$$

Equation (1) tensored with $\otimes_{A\oplus M} A$ gives the following sequence in Mod_A:

$$L \otimes_{A\oplus M} M \otimes_{A\oplus M} A \to L \otimes_{A\oplus M} A \xrightarrow{s_L} L \otimes_{A\oplus M} A \otimes_{A\oplus M} A.$$

Also Equation (1) gives a map

$$\gamma_L : L \otimes_{A\oplus M} A \to L \otimes_{A\oplus M} M[1],$$

in $\mathrm{Mod}_{A\oplus M}$. By adjunction this gives a map:

$$\gamma_L^A : L \otimes_{A\oplus M} A \otimes_{A\oplus M} A \to L \otimes_{A\oplus M} A \otimes_A M[1]$$

in Mod_A. Hence one can define the map

$$\sigma : Q \to L \otimes_A M[1],$$

by $\sigma = (\eta_L \otimes \mathrm{id}_A) \circ \gamma_L^A \circ s_L \circ \imath$.

Now suppose given $N \in \mathrm{Mod}_{A\oplus M}$ such that $N \otimes_{A\oplus M} A \simeq L$. Considering the following sequence

$$N \otimes_{A\oplus M} M \to N \to N \otimes_{A\oplus M} A$$

of $A \oplus M$-modules. This is equivalent to

$$L \otimes_A M \to N \to L,$$

which gives a map $\gamma_N^A : L \otimes_{A\oplus M} A \to L \otimes_A M[1]$.

Lemma 2. *The map σ defined before coincides with the composite $\gamma_N^A \circ \imath$.*

It is enough to check: $\gamma_L^A = (\eta_L \otimes \mathrm{id}_A) \circ \gamma_L^A \circ s_L$ which follows from:

(i) $s_L \circ (\eta_L \otimes \mathrm{id}_A) : L \otimes_{A\oplus M} A \otimes_{A\oplus M} A \to L \otimes_{A\oplus M} A \otimes_{A\oplus M} A$ is the identity;

(ii) $\gamma_N^A \circ (\eta_L \otimes \mathrm{id}_A) = (\eta_L \otimes \mathrm{id}_A) \circ \gamma_L^A$.

For (i) one considers the section $s_N : L \to L \otimes_{A\oplus M} A$ tensored with $\otimes_{A\oplus M} A$

$$s_N \otimes \mathrm{id}_A : L \otimes_{A\oplus M} A \to L \otimes_{A\oplus M} A \otimes_{A\oplus M} A,$$

which is just s_L.

Item (ii) follows from the commutativity of the diagram below, where the lefthand square

$$\begin{array}{ccccc}
L \otimes_{A\oplus M} A & \xrightarrow{s_L} & L \otimes_{A\oplus M} A \otimes_{A\oplus M} A & \xrightarrow{\eta_L \otimes \mathrm{id}_A} & L \otimes_{A\oplus M} A \\
\downarrow{\gamma^A}_N & & \downarrow{\gamma^A}_L & & \downarrow{\gamma^A}_N \\
L \otimes_A M[1] & \xrightarrow[s_N \otimes \mathrm{id}_A]{} & L \otimes_{A\oplus M} A \otimes_A M[1] & \xrightarrow[\eta_L \otimes \mathrm{id}_A]{} & L \otimes_A M[1]
\end{array}$$

commutes because the map between the kernels of γ_N^A and γ_L^A is just s_N. Namely, the commutativity of the following diagram:

$$\begin{array}{ccccc}
L \otimes_{A\oplus M} A & \longrightarrow & L \otimes_{A\oplus M} A \otimes_{A\oplus M} A & \xrightarrow{{\gamma^A}_L} & L \otimes_{A\oplus M} A \otimes_A M[1] \\
\uparrow s_N & & \uparrow s_L & & \uparrow s_N \otimes \mathrm{id}_A \\
L & \longrightarrow & L \otimes_{A\oplus M} A & \xrightarrow[{\gamma^A}_N]{} & L \otimes_A M[1]
\end{array}$$

This proves the following.

Proposition 9. *For a fixed $L \in Mod_A$, the space*

$$\{N \in Mod_{A\oplus M}; | \; N \otimes_{A\oplus M} A \simeq L\}$$

is isomorphic to $Hom_{A\oplus M}(L \otimes_{A\oplus M} A, L \otimes_A M[1])$.

Corollary 3. *For Mod_A^{aperf} the category of almost perfect complexes of A-modules one has:*

$$T_L^* Mod_A^{\mathrm{aperf}} \simeq L \otimes L^\vee[-1].$$

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