NEWTON DECOMPOSITION ON THE QUOTIENT STACKS OF LOOP GROUPS

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Abstract. In this paper, we initiate the study of affine character sheaves. We start by considering $D(\frac{LG}{LG})$ an approriately defined category of étale ℓ-adic focus on the quotient stack $\frac{LG}{LG}$ of a loop group LG by its conjugation action. The first main result is a decomposition of $\frac{LG}{LG}$ into locally closed finitely presented substacks corresponding to Newton strata, which gives a semi-orthogonal decomposition of the $D(\frac{LG}{LG})$. Our second main result is a realization of the categorical cocenter of the affine Hecke category $D(\mathrm{Iw}\backslash LG/\mathrm{Iw})$, i.e. the category of unipotent affine character sheaves, as a full subcategory of $D(\frac{LG}{LG})$. We finish by constructing a semi-orthogonal decomposition of the category of unipotent character sheaves using the Newton strata, this provides a categorification of the Newton decomposition on the cocenter of the Hecke algebra of the p-adic groups, as established in [12]. Our results hold both in the mixed and equal characteristic set up.

Contents

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Introduction

0.1. **Lusztig's Theory of Character Sheaves.** We first give a brief review of Lusztig's theory of character sheaves on reductive groups. Let $\mathbf{G}$ be a connected reductive group over an algebraically closed field $\mathbf{k}$. Let G be a group scheme over $\mathbf{k}$, such that $G = \mathbf{G}(\mathbf{k})$ and pick a maximal torus and Borel $T \subset B \subset G$ subgroup schemes. One has $B = TU$, where U is an unipotent group scheme. We have a Bruhat stratification $G = \sqcup_{w \in W} B\dot{w}B$, where each $B\dot{w}B \hookrightarrow G$ is a locally closed subscheme and W is the Weyl group of G. We then have natural maps of quotient stacks:

$$B\backslash G/B \xleftarrow{q} \tfrac{G}{B} \xrightarrow{p} \tfrac{G}{G},$$

where $\frac{A}{H}$ denotes the quotient stack of A under the conjugation action of the group H. The character functor is given by

$$CH := p_! \circ q^* : D\left(B\backslash G/B\right) \longrightarrow D\left(\frac{G}{G}\right),$$

where $D(-)$ is the derived category of constructible sheaves on the quotient stacks. The category of (unipotent) character sheaves is the full subcategory of $D\left(\frac{G}{G}\right)$ generated by the essential image of CH under colimits.

It was observed by Ben–Zvi and Nadler in [4] that for k the complex numbers and considering D-modules as the sheaf theory, the category of unipotent character sheaves is the categorical trace of $D(B\backslash G/B)$. At the level of abelian categories this was first proved in [6].

When k is a field of positive characteristic, for instance $k = \overline{\mathbb{F}}_q$ then one can relate character sheaves on G to character of the group $G(\mathbb{F}_q)$. Indeed, we consider the theory of étale constructible sheaves on G and let $\sigma : G \to G$ be the Frobenius morphism. Given $\mathcal{F}$ a character sheaf on G with $\sigma^*\mathcal{F} \cong \mathcal{F}$. The trace of Frobenius on the stalk of $\mathcal{F}$ over $g \in G^\sigma = \mathbf{G}(\mathbb{F}_q)$ yields a characteristic function on the finite group $\mathbf{G}(\mathbb{F}_q)$. In fact, all the almost characters of $\mathbf{G}(\mathbb{F}_q)$ can be obtained in this manner.

In the case of étale ℓ-adic sheaves it was initially proved in [30] that character sheaves are obtained as the categorical trace of the abelian category of perverse sheaves on $B\backslash G/B$ and the statement for the stable ∞-categories of derived sheaves was proved in [21], building on the argument of [4].

Our ultimate goal is to generalize Lusztig's theory of character sheaves to loop groups. In this paper, we study the categorical trace (co-center) of the affine Hecke category, which is expected to realize the derived category of unipotent affine character sheaves. We realize the categorical trace as a subcategory of étale constructible sheaves on the loop group equivariant with respect to the conjugation action.

The point of view that we will adopt is to define the derived category of (unipotent) character sheaves as the categorical trace of Iwahori-equivariant étale constructible sheaves on the affine flag variety. We will realize such defined category of (unipotent) character sheaves as a full subcategory of the category of étale constructible sheaves on the loop group equivariant with respect to the conjugation action.

0.2. **Newton Decompositions.** Given a perfect field k, let $L = k((\epsilon))$ or L is a non-archimedean local field whose residue field is k. We consider a connected reductive group $\mathbf{G}$ over the field L, where k is an algebraically closed field. Let LG be the loop group ind-scheme, its k-points are given by $\mathbf{G}(L)$. This is actually an ind-perfect scheme in the mixed characteristic case, we will ignore this point in the introduction see §2.1 for details.

Consider $\theta : LG \to LG$ an automorphism of LG and let $\frac{LG}{Ad_\theta(LG)}$ denote the quotient stack by the θ-twisted conjugation action, i.e. $(g, h) \mapsto gh\theta(g)^{-1}$. There are two main cases of interest: θ is the identity and θ is the Frobenius, when k is of characteritstic p.

The quotient stack $\frac{LG}{Ad_\theta(LG)}$ is a very complicated object to study. The first problem is that LG is not an ind-finite type scheme the second is that we are taking the quotient by LG itself. Many results can be proved by relaxing either of these difficulties:

- in [7] one consider the quotient $\frac{\mathfrak{C}}{Ad(LG)}$, where $\mathfrak{C}$ is the comptact part of the Lie algebra of LG, which is a placid scheme (see §1.4.4);
- if one considers LG/Iw or $\mathrm{Iw}\backslash LG/\mathrm{Iw}$ for Iw the Iwahori subgroup scheme, which is a placid scheme, then many results are proved for the categories $D(LG/\mathrm{Iw})$ and $D(\mathrm{Iw}\backslash LG/\mathrm{Iw})$ (e.g. [5]).

Before stating our results we review what was known at the level of k-points about the quotient stack $\frac{LG}{Ad_\theta(LG)}$. For $L = \overline{\mathbf{Q}}_p$ and $\theta = \sigma$ the Frobenius automorphism of $\breve{G}$, in [23, 24], Kottwitz classified the set $B(G)$ of σ-conjugacy classes of $\breve{G}$ by using the Kottwitz map and Newton polygons. In [12], the first author established a bijection $B(G) \simeq \breve{W} /\!\!/_\sigma \breve{W}$, where $\breve{W} /\!\!/_\sigma \breve{W}$ is a subset of special σ-twisted conjugacy classes of $\breve{W}$ (see §2.1.2 for a precise definition) and for each point $\mathcal{O} \in \breve{W} /\!\!/_\sigma \breve{W}$ defined a locally closed (admissible) subset $\breve{G}_\mathcal{O} \subset \breve{G}$ stable under $\breve{G}$ conjugation, such that $\breve{G} = \sqcup_{\breve{W} /\!\!/_\sigma \breve{W}} \breve{G}_\mathcal{O}$.

In the first author's joint work with Nie [16], these results were generalized to $L = k((\epsilon))$ and θ is any automorphism of $\breve{G}$ determined by a length-preserving automorphism of $\breve{W}$. That is one has a decomposition:

$$\breve{G} \simeq \sqcup_{\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}} \breve{G}_\mathcal{O}, \tag{0.1}$$

where each $\breve{G}_\mathcal{O}$ is a $\breve{G}$-conjugation stable locally closed admissible (see §2.1.2 for an explanation of this terminology) subset.

0.3. **Schematic decomposition of $\frac{LG}{LG}$.** In the rest of the introduction, for simplicity, we only discuss the case where θ is the identity map and we write $\frac{LG}{LG}$ instead of $\frac{LG}{\mathrm{Ad}(LG)}$. In the main context, we also consider the nontrival group automorphisms.

The first main result of this paper is a lift of the decomposition (0.1) to the level of ind-schemes:

Theorem 0.1. *For each Newton point $\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}$ there is a reduced ind-scheme $LG_{\mathcal{O}}$ equipped with a finitely presented locally closed embedding $\jmath_{\mathcal{O}} : LG_{\mathcal{O}} \hookrightarrow LG$, such that $LG_{\mathcal{O}}(\mathbf{k}) \simeq \breve{G}_{\mathcal{O}}$.*

The schematic image is always closed and thus it is difficult to define a locally closed subscheme without using its closure. Thus despite the notation $LG_{\overline{\mathcal{O}}}$, we need to define $LG_{\overline{\mathcal{O}}}$ before introducing $LG_{\mathcal{O}}$. This definition is more involved than the definition of $\breve{G}_{\mathcal{O}}$ even set-theoretically.

The definition consists in picking a representative $w_{\mathcal{O}} \in \breve{W}$ and studying the conjugation action of a truncation of LG on the Schubert variety corresponding to $w_{\mathcal{O}}$. Using the Deligne–Lusztig reduction method of [13], we can reduced the analysis to the straight elements in $\breve{W}$. In this case the conjugation action can be "locally" factored through a fp quotient and the schematic image stabilizes when passing from local subspaces of LG to the whole LG. This leads to the schematic definition of $LG_{\overline{\mathcal{O}}}$. One defines $LG_{\mathcal{O}}$ as the complement of the smaller closed strata. Finally we check that the the open part does not depend on the choice of $w_{\mathcal{O}}$, which implies that the closed stratum is also independent of $w_{\mathcal{O}}$.

0.4. **Decomposition of $D(\frac{LG}{LG})$.** At this point it is important to discuss the sheaf theory considered in this paper. There are many obstacles to defining a good enough sheaf theory on the stack $\frac{LG}{LG}$. The first is that LG itself is not an ind-scheme of finite type. Thus, one needs to consider sheaves on scheme not of finite type. However, one can present LG as a filtered colimit of placid schemes, these are schemes possibly of infinite type on which one still has a good theory of sheaves. The notion of placid scheme was previously considered in [33, 7], but their definition is not enough for the sheaf formalism that we need. The definition of placid scheme we use is the more general notion introduced by Hemo and the third author in [20]. The main point is that when restricted to placid stacks the sheaf theory recovers some of the coherent six-functor formalism, that was lost on schemes not of finite type.

The second problem is how to take the quotient of LG by itself. A sheaf theory on the quotient $\frac{LG}{LG}$ can be defined as the right Kan extension of sheaves on all schemes, i.e. constructible sheaves on $\frac{LG}{LG}$ are !-pullback compatible collections of sheaves on each scheme mapping to it. In this set up one can have $*$-pushforward and !-pullback glueing for fp closed embedding. However, to formulate a semi-orthogonal decomposition one wants to perform !-pushforward and !-pullback glueing. This is achieved by restricting to placid stacks, i.e. stacks that admit a cover by placid schemes. Then, the open-closed for the sheaf theory on placid stacks, ind-placid stacks or sifted-placid stacks, reduces to the open-closed glueing of sheaves on placid schemes, which has a $*$-pushforward and pullback and !-pushforward and pullback open-closed glueing for finitely presented embeddings.

After the sheaf theory on $\frac{LG}{LG}$ is established, we obtain our second main result, which is a categorical version of Theorem 0.1:

Theorem 0.2. *One has a semi-orthogonal decomposition of $D\left(\frac{LG}{LG}\right)$ indexed by $\breve{W} /\!\!/ \breve{W}$, whose strata is given by $D\left(\frac{LG_{\mathcal{O}}}{LG}\right)$.*

The result Theorem 0.2 follows formally from Theorem 0.1. However, we notice that since the set $\breve{W} /\!\!/ \breve{W}$ is not totally ordered the notion of semi-orthogonal decomposition is more subtle than usual definitions in the literature, e.g. [1]. In the appendix, we provide a summary of the theory of stratifications (aka semi-orthogonal decompositions) of stable ∞-categories as developed in [3] but adapted to the context of constructible sheaves on infinite-dimensional objects.

0.5. **Categorical trace of affine Hecke category.** Our last result is:

Theorem 0.3. *Let* Iw *be the Iwahori group scheme determined by the choice of an Iwahori subgroup of* $\mathbf{G}$*. Consider the affine Hecke category* $D(\mathrm{Iw}\backslash LG/\mathrm{Iw})$*.*

(1) The categorical trace $\mathrm{Tr}(D(\mathrm{Iw}\backslash LG/\mathrm{Iw}))$ *is a full subcategory of* $D\left(\frac{LG}{LG}\right)$*, generated under colimits by the essential image of* $CH := p_* \circ q^!$ *from* $D(\mathrm{Iw}\backslash LG/\mathrm{Iw})$ *to* $D\left(\frac{LG}{LG}\right)$ *defined via pull-push with respect to:*

$$\mathrm{Iw}\backslash LG/\mathrm{Iw} \xleftarrow{q} \frac{LG}{\mathrm{Iw}} \xrightarrow{p} \frac{LG}{LG}\ .$$

(2) There is a semi-orthogonal decomposition of $\mathrm{Tr}(D(\mathrm{Iw}\backslash LG/\mathrm{Iw}))$ *indexed by* $\breve{W} /\!\!/ \breve{W}$*, whose strata is given by* $\mathrm{Tr}(D(\mathrm{Iw}\backslash LG/\mathrm{Iw})) \cap (i_{\mathcal{O}})_!\left(D\left(\frac{LG_{\mathcal{O}}}{LG}\right)\right)$*, and is generated under colimits by the restriction of* CH *to the subcategory* $D(Iw\backslash LG_w/Iw)$ *for certain elements* $w \in \breve{W}$*.*

For the affine Hecke algebra H, it was proved in [17] that the cocenter $\bar{H}$ has a standard basis $\{T_{\mathcal{O}}\}$, where $\mathcal{O}$ runs over conjugacy classes of $\tilde{W}$. This leads to the Newton decomposition of the cocenter of affine Hecke algebra

$$\bar{H} = \oplus \bar{H}_\nu, \text{ where } H_\nu \text{ is spanned by the image of } T_w, \text{ for certain } w \in \tilde{W}.$$

Part (2) of Theorem gives a categorification of the above result. Our semi-orthogonal decomposition is also a refinement of the decomposition obtained in [25]. In the case where θ is the Frobenius automorphism, the analogue of Theorem 0.3 was obtained by Hemo and the third author in [20].

To compute the categorical trace, one may apply a general theorem of Lurie's that allows one to concretely compute the colimit of a simplicial diagram of ∞-categories. The procedure is to find an augmentation of the simplicial diagram such that all the connecting functors in the augmented diagram admit coherent right adjoints. One way to actually realize this strategy is to use geometry, namely to compute the so-called geometric trace (cf. [20, §7]). In our situation, the augmentation is provided by the category $D\left(\frac{LG}{LG}\right)$ and the character functor $CH = p_* q^! := D(\mathrm{Iw}\backslash LG/\mathrm{Iw}) \to D\left(\frac{LG}{LG}\right)$. However, to understand the augmented diagram, we also need right adjoints to the functors $q^!$ and p_*. The functor $q^!$ that causes problem, since its fibers are not representable.

Thus, we consider an ind-finite version of sheaves, i.e. ind-extension of constructible sheaves on placid stacks, which are defined by descent on a cover by placid schemes (see §1.5.3). In the context of ind-finite sheaves on $\frac{LG}{\mathrm{Iw}}$ and $\mathrm{Iw}\backslash LG/\mathrm{Iw}$, one has an adjunction $(q^!, q_*^{\mathrm{ren}})$ and the geometric trace of $D(\mathrm{Iw}\backslash LG/\mathrm{Iw})$ agrees with the categorical trace. Finally to prove part (2), we use the specifics of the definition of $D(LG_{\mathcal{O}})$ to check that the Newton decomposition of $D\left(\frac{LG}{LG}\right)$ is compatible with the cocenter $\mathrm{Tr}(\mathcal{H})$, and the (upgraded version of) Deligne-Lusztig reduction on the sheaf in $D\left(\frac{LG}{LG}\right)$.

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1. Constructible sheaves

1.1. **Conventions and notation.**

(a) Let F denote a discretely valued (complete) local field, $\mathcal{O}_F \subset F$ its ring of integers, $\mathfrak{m} \subset \mathcal{O}_F$ its maximal ideal and $k := \mathcal{O}_F/\mathfrak{m}$ the residue field, which we assume is algebraically closed.

(b) Let Pr denote the ∞-category of presentable stable ∞-categories with morphisms exact functors. Let Pr^{L} (resp. Pr^{R}) denotes the subcategory of Pr where morphisms are left (resp. right) adjoints. The category Pr has limits and colimits ([10, Chapter 1, Corollary 5.3.4]) and Pr^{L} (resp. Pr^{R}) is closed under limits (resp. colimits). One has an equivalence $\mathrm{Pr}^{\mathrm{L}} \xrightarrow{\sim} (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}$ given by passing to the right adjoints ([29, Corollary 5.5.3.4]).

(c) Colimits in Pr^{L} can be computed as follows. Given a small diagram $\mathcal{C}_\bullet : I \to \mathrm{Pr}^{\mathrm{L}}$, let $\mathcal{C}_\bullet^{\mathrm{R}} : I^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}} \hookrightarrow \mathrm{Pr}$ denote the diagram obtained by passing to right adjoints. One has $\mathrm{colim}_I\, \mathcal{C}_i \xrightarrow{\simeq} \lim_{I^{\mathrm{op}}} \mathcal{C}_i^{\mathrm{R}}$, where the colimit is taken in Pr^{L} and the limit in Pr ([10, Chapter 1, Proposition 2.5.7] or [29, Corollary 5.5.3.4]).

(d) Let ℓ be a natural number coprime to the characteristic of k. The ring of coefficients E for the sheaf theories that we consider in this article can be very general, for instance we can consider $E = \mathbb{F}_\ell, \mathbb{Z}_\ell$, or $\overline{\mathbb{Q}_\ell}$ ([20, §10.2.1] for the detailed assumptions). For a choice of E we let Mod_E be the derived ∞-category of complexes of E-modules.

(e) The Lurie tensor product ([28, §4.8.1]) endows Pr^{L} with the structure of a symmetric monoidal ∞-category. The stable ∞-category Mod_E is a commutative algebra object in Pr^{L} ([28, Theorem 4.5.2.1 and Theorem 7.1.2.13]).

(f) Let Lincat_E denote the ∞-category of Mod_E-modules in Pr^{L}. We also let $\mathrm{Lincat}_E^{\mathrm{c.g.}}$ denote the subcategory of Lincat_E whose objects are compactly generated ∞-categories and morphism are compact object preserving functors.

(g) We also need the small variant of (f). Let $\mathrm{Lincat}_E^{\mathrm{perf}}$ denote the ∞-category of small idempotent complete E-linear stable ∞-categories with morphisms exact functors.

(h) The contexts (f) and (g) are related via the equivalence

$$\mathrm{Ind} : \mathrm{Lincat}_E^{\mathrm{perf}} \rightleftarrows \mathrm{Lincat}_E^{\mathrm{c.g.}} : (-)^\omega \ ,$$

where Ind is the construction of formally adding all filtered colimits and $(-)^\omega$ passes to the subcategory of compact objects.

(i) One has self-dualities $\mathrm{Lincat}_E^{\mathrm{c.g.}} \xrightarrow{\sim} \mathrm{Lincat}_E^{\mathrm{c.g.}}$ sending $\mathcal{C}$ to $\mathcal{C}^\vee$ and similarly $\mathrm{Lincat}_E^{\mathrm{perf}} \xrightarrow{\sim} \mathrm{Lincat}_E^{\mathrm{perf}}$ sending $\mathcal{C}_0$ to $\mathcal{C}_0^{\mathrm{op}}$. These are related by (h), given $\mathcal{C}_0 \in \mathrm{Lincat}_E^{\mathrm{perf}}$ one has $\mathrm{Ind}(\mathcal{C}_0)^\vee \xrightarrow{\sim} \mathrm{Ind}(\mathcal{C}_0^{\mathrm{op}})$.

(j) When we have a pair of functors between two categories, unless otherwise stated we will always write the left adjoint functors on top of the right adjoint.

(k) Given a commutative diagram of ∞-categories:

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "F'"] \arrow[d, "G_{\mathcal{C}}"'] & \mathcal{D}' \arrow[d, "G_{\mathcal{D}}"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd} \tag{1.1}$$

we will say (1.1) is *horizontally left* (resp. *right*) *adjointable* if F' and F admit left (resp. right) adjoints $(F')^{\mathrm{L}}$ and F^{L} (resp. F'^{R} and F^{R}) and the canonical morphism:

$$F^{\mathrm{L}} \circ G_{\mathcal{D}} \longrightarrow G_{\mathcal{C}} \circ (F')^{\mathrm{L}} \quad \big(\text{resp. } G_{\mathcal{C}} \circ (F')^{\mathrm{R}} \longrightarrow F^{\mathrm{R}} \circ G_{\mathcal{D}}\big)$$

is an equivalence. We will say that (1.1) is *vertically left* (resp. *right*) *adjointable* if the transposed diagram is horizontally left (resp. right) adjointable.

1.2. **Review of geometric objects.** In this subsection, we review some algebro-geometric objects on which we will consider the theory of constructible ℓ-adic sheaves.

1.2.1. *Prestacks.* Let Aff be the category of (classical) affine k-schemes. Let Sch denote the category of quasi-compact quasi-separated (qcqs) k-schemes and $\mathrm{Sch}_{\mathrm{ft}}$ the full subcategory of schemes of finite type. Let AlgSpc denote the category of qcqs algebraic spaces and $\mathrm{AlgSpc}_{\mathrm{ft}}$ the full subcategory of algebraic spaces of finite type.

A *prestack* is an accessible functor $\mathscr{X} : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Spc}$ from the opposite of the category of affine schemes to the ∞-category of anima. We let PStk denote the category of prestacks. A *stack* is a prestack that satisfies descent with respect to the étale topology on affine schemes. A *k-space* is a stack $\mathscr{X}$ such that for every affine scheme S, the space $\mathscr{X}(S)$ is discrete, i.e., all its positive homotopy groups vanish.

Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism between prestacks. We say that f is *affine* (resp. *schematic*, *representable*, *open*, *closed*, *locally closed*) if for every $S \to \mathscr{Y}$, where S is an affine scheme, the base change $\mathscr{X} \underset{\mathscr{Y}}{\times} S$ is an affine scheme (resp. qcqs scheme, qcqs algebraic space, an open embedding, a closed embedding, a locally closed embedding)[1]. We say that f is *fp-affine* (resp. *fp-schematic*, *fp-representable*, *fp-open*, *fp-closed*, *fp-locally closed*) if in addition $\mathscr{X} \underset{\mathscr{Y}}{\times} S \to S$ is finitely presented.

1.2.2. *Ind-schemes.* A k-space $\mathscr{X}$ is said to be an *ind-scheme* (resp. *ind-algebraic space*) if it admits a *presentation* $\mathrm{colim}_I X_i \xrightarrow{\sim} \mathscr{X}$, where I is a filtered diagram, each X_i is a qcqs scheme (resp. qcqs algebraic space), and for each $i \to j$ the morphism $X_i \to X_j$ is a finitely presented closed embedding[2].

An ind-scheme (resp. ind-algebraic space) $\mathscr{X} \simeq \mathrm{colim}_I X_i$ is said to be *ind-fp* if each $X_i \in \mathrm{Sch}_{\mathrm{ft}}$ (resp. $\mathrm{AlgSpc}_{\mathrm{ft}}$). An ind-scheme (resp. ind-algebraic space) $\mathscr{X} \simeq \mathrm{colim}_I X_i$ is said to be *ind-fp-proper* if each X_i is proper.

Similarly to §1.2.1, we say that a morphism $f : \mathscr{X} \to \mathscr{Y}$ between prestacks is *ind-schematic* (resp. *ind-representable*) if for every $S \to \mathscr{Y}$ where S is an affine scheme the base

[1] Note that as we impose the qcqs condition, the notion of schematic (resp. representable) morphisms considered in this article is slightly more restrictive than the same named notion in literature.

[2] These are known as reasonable ind-schemes (or ind-algebraic spaces) (e.g. [33, §6.7]) and are more restrictive than general ind-schemes (or ind-algebraic spaces), but they will be enough for our purposes.

change $\mathscr{X} \underset{\mathscr{Y}}{\times} S$ is an ind-scheme (resp. ind-algebraic space). Furthermore, an ind-schematic or ind-representable morphism $f : \mathscr{X} \to \mathscr{Y}$ is *ind-fp* (resp. *ind-fp-proper*) if every fiber $\mathscr{X} \underset{\mathscr{Y}}{\times} S$ is ind-fp (resp. ind-fp-proper).

1.2.3. *Variant: Perfect geometry.* In the mixed case, i.e. F has characteristic 0 and char $k = p$ we have the following variation of the previous geometric objects (see [37, §A.1]).

Let $\mathrm{Aff}^{\mathrm{perf}}$ denote the opposite of the category of perfect k-algebras. In this setting a *perfect prestack* is a functor $\mathscr{X} : \mathrm{Aff}^{\mathrm{perf}} \to \mathrm{Spc}$ and a *perfect k-space* is stack with respect to the fpqc topology. The restriction along $\mathrm{Aff}^{\mathrm{perf,op}} \hookrightarrow \mathrm{Aff}^{\mathrm{op}}$ gives a functor

$$(-)_{\mathrm{perf}} : \mathrm{Fun}(\mathrm{Aff}^{\mathrm{op}}, \mathrm{Spc}) \longrightarrow \mathrm{Fun}(\mathrm{Aff}^{\mathrm{perf,op}}, \mathrm{Spc})$$

called the *perfection* functor.

Following [36, Appendix A], we let $\mathrm{Sch}^{\mathrm{perf}}$ denote the category of qc qs *perfect k-schemes*, i.e. perfect k-spaces that satisfy Zariski descent and admit a Zariski atlas. We let $\mathrm{AlgSpc}^{\mathrm{perf}}$ denote the category of qc qs *perfect algebraic spaces*, i.e. perfect k-spaces that satisfy étale descent, have (perfect) schematic diagonal and admit an étale surjection from a perfect scheme. By [37, Lemma A.12] the composition $\mathrm{Sch}_{\mathrm{perf}} \hookrightarrow \mathrm{Sch} \overset{(-)_{\mathrm{perf}}}{\to} \mathrm{Sch}^{\mathrm{perf}}$ (resp. $\mathrm{AlgSpc}_{\mathrm{perf}} \hookrightarrow \mathrm{AlgSpc} \overset{(-)_{\mathrm{perf}}}{\to} \mathrm{AlgSpc}^{\mathrm{perf}}$) is an equivalence, where $\mathrm{Sch}_{\mathrm{perf}}$ (resp. $\mathrm{AlgSpc}_{\mathrm{perf}}$) is the subcategory of k-schemes (resp. algebraic spaces) X such that the Frobenius morphism $\phi_X : X \xrightarrow{\sim} X$ is an isomorphism. Implicitly using this equivalence we let:

$$\imath_{\mathrm{perf}} : \mathrm{Sch}^{\mathrm{perf}} \hookrightarrow \mathrm{Sch} \qquad \imath_{\mathrm{perf}} : \mathrm{AlgSpc}^{\mathrm{perf}} \hookrightarrow \mathrm{AlgSpc} \tag{1.2}$$

denote the inclusion of perfect qc qs schemes (resp. algebraic spaces) into qc qs schemes (resp. algebraic spaces). One has adjunctions $(\imath_{\mathrm{perf}}, (-)_{\mathrm{perf}})$.

Recall a morphism $f : S \to T$ between perfect affine schemes is said to be perfectly finitely presented, if there exist $\tilde{f} : \tilde{S} \to \tilde{T}$ a finitely presented morphism between affine schemes, such that $\tilde{f}_{\mathrm{perf}} = f$. Thus, we have the definitions completely analogous to §1.2.1 and §1.2.2 by replacing finitely presented (fp) everywhere by perfectly fintiely presented (pfp).

1.3. **Sheaf formalism: general setup.** The amount of data encoded by the different sheaf formalisms that we need in this article can be understood in three instances:

(1) a 3-functor formalism on the most general geometric objects, e.g. sheaves on *prestacks* with arbitrary !-pullback, and $*$-*pushforward for representable ind-fp morphisms*;
(2) a 3-functor formalism on *sifted placid stacks* where one has left adjoints for !-pullback along representable ind-fp morphisms and left adjoints for $*$-pushfoward along representable fp morphisms;
(3) a 3-functor formalism on *ind-placid stacks* where one has right adjoints to !-pullbacks with respect to certain *non-representable* morphisms (weakly cohomologically pro-smooth) with enough base change.

1.3.1. *3-functor formalism.* Now we introduce a couple of abstract definitions to conveniently express the iterations of the sheaf formalisms and perform the extension steps that we need. Let $\mathcal{C}$ be a category of geometric objects, e.g. $\mathcal{C} = \mathrm{Sch}_{\mathrm{ft}}$ or $\mathcal{C} = \mathrm{PStk}$.

Definition 1.1. Let $\mathcal{C}$ be a category and consider two classes of morphisms E_l, E_r, such that pullbacks of the form (1.6) exists, whenever $g_Y \in E_r$ and $f \in E_l$. We also assume that E_l and E_r contain all isomorphisms and are stable under compositions and pullback. The category $\mathrm{Corr}(\mathcal{C})_{E_l;E_r}$ has as:

- objects the same as objects in $\mathcal{C}$;
- morphisms $X \rightsquigarrow Y$ given by correspondences $X \xleftarrow{f} Z \xrightarrow{g} Y$, where $f \in E_l$ and $g \in E_r$.

The composition of morphisms is given by taking pullbacks. Notice that the conditions on E_l and E_r guarantee that we have identity morphisms and that compositions are well-defined. When we take $E_l = E_r$ to contain all morphisms of $\mathcal{C}$ we write $\mathrm{Corr}(\mathcal{C})$.

We also assume that $\mathcal{C}$ admits finite products, thus one has a (Cartesian) symmetric monoidal structure on $\mathcal{C}$ which induces a symmetric monoidal structure on $\mathrm{Corr}(\mathcal{C})$.

Following [10, Chapter 5, Introduction], [34, Definition 3.1] and [31, Definition A.5.7] we make the following:

Definition 1.2. A 3-*functor formalism* on $\mathcal{C}$ is a lax symmetric monoidal functor

$$D : \mathrm{Corr}(\mathcal{C})_{E_l,E_r} \longrightarrow \mathrm{Lincat}^{?}_{E}, \tag{1.3}$$

where $? \in \{\emptyset, \text{c.g.}, \mathrm{Perf}\}$.

Concretely, this encodes the following data:

- for every $Y \xleftarrow{f} X \xrightarrow{\mathrm{id}_X} X$ with $f \in E_l$ a !-pullback functor $f^! : D(Y) \to D(X)$;
- for every $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{g} Y$ with $g \in E_r$ a $*$-pushforward functor $g_* : D(X) \to D(Y)$;
- for every $X \in \mathcal{C}$ an exterior tensor product $\boxtimes : D(X) \otimes D(X) \to D(X \times X)$.

Notice that for every $X \in \mathcal{C}$ such that $\Delta_X : X \to X \times X$ belongs to E_l, we have a tensor product $(-) \otimes (-) : D(X) \otimes D(X) \xrightarrow{\boxtimes} D(X \times X) \xrightarrow{\Delta^!_X} D(X)$.

1.3.2. *Extra adjunctions.* The formalism encoded in Definition 1.2 is neat but we need two extra things: (1) explicit relations between f_* and $f^!$ in certain situations, this is relevant both in constructing such formalism but it also encodes the open-closed gluing exact sequences and (2) we will need left adjoints for $f^!$ and g_* in certain situations with enough base change properties. Thus, we introduce the following concepts.

Definition 1.3. Let $D : \mathrm{Corr}(\mathcal{C})_{E_l,E_r} \to \mathrm{Lincat}^{?}_{E}$ be a 3-functor formalism.

(1) A class of weakly stable morphisms $E^{\mathrm{L}}_l \subset E_l$ is said to be *left-adjointable* if for every $f : X \to Y \in E^{\mathrm{R}}_l$ the functor $f^!$ admits a left adjoint $f_\dagger$. Moreover, we say that E^{L}_l
  - (i) is *compatible with* $E^{\mathrm{BC}}_l \subset E_l$ *base change* if $(f')_\dagger \circ g^!_X \xrightarrow{\sim} g^!_Y \circ f_\dagger$ for every $g_Y \in E^{\mathrm{BC}}_l$;
  - (ii) is *compatible with* $E^{\mathrm{PC}}_r \subset E_r$ *pushforward* if $f_\dagger \circ (g_X)_* \xrightarrow{\sim} (g_Y)_* \circ f'_\dagger$ for every $g_Y \in E^{\mathrm{PC}}_r$;
  - (iii) *satisfies projection formula* if $f_\dagger(\mathscr{F} \otimes f^!(\mathscr{G})) \xrightarrow{\sim} f_\dagger(\mathscr{F}) \otimes \mathscr{G}$ for every $\mathscr{F} \in D(X)$ and $\mathscr{G} \in D(Y)$.

(2) A class of weakly stable morphisms $E^{\mathrm{L}}_r \subset E_r$ is said to be *left-adjointable* if for every $f : X \to Y \in E^{\mathrm{L}}_r$ the functor f_* admits a left adjoint $f^\star$. Moreover, we say that E^{L}_r
  - (i) is *compatible with* $E^{\mathrm{BC}}_r \subset E_r$ *base change* if $f^\star \circ (g_Y)_* \xrightarrow{\sim} (g_X)_* \circ (f')^\star$ for every $g_Y \in E^{\mathrm{BC}}_r$;
  - (ii) is *compatible with* $E^{\mathrm{PC}}_l \subset E_l$ *pullback* if $(f')^\star \circ g^!_Y \xrightarrow{\sim} g^!_X \circ f^\star$ for every $g_Y \in E^{\mathrm{PC}}_l$;

(iii) *satisfies projection formula* if $f_*(\mathscr{F})\otimes\mathscr{G}\xrightarrow{\sim} f_*(\mathscr{F}\otimes f^\star(\mathscr{G}))$ for every $\mathscr{F}\in D(X)$ and $\mathscr{G}\in D(Y)$.

Remark 1.4. (1) In Definition 1.3 (1) we say that $E_l^{\mathrm{L}}\subset E_l$ is *internally left-adjointable*, if $(f_*, f^!)$ is an adjunction. Notice that given $E_l^{\mathrm{L},1}\subset E_l$ internally left-adjointable and $E^{\mathrm{L},2}\subset E_l$ left adjointable we obtain an isomorphism $f_\dagger\xrightarrow{\sim} f_*$[3], whenever $f\in E_l^{\mathrm{L},1}\cap E_l^{\mathrm{L},2}$. We have a similar notion for Definition 1.3 (2) with $f^\star\xrightarrow{\sim} f^!$.

(2) In Definition 1.3 (1) if $E_l^{\mathrm{BC}}=E_l$ or $E_r^{\mathrm{PC}}=E_r$ we will omit the mention of compatibility with base change or pushforward. In particular, if $E_l^{\mathrm{BC}}=E_l$ and $E_r^{\mathrm{PC}}=E_r$ we will simply say that the class E_l^{L} is nicely left adjointable. Similarly for Definition 1.3 (2) if $E_r^{\mathrm{BC}}=E_r$ and/or $E_l^{\mathrm{PC}}=E_l$.

(3) Notice that when $(X\overset{\Delta_X}{\to} X\times X)\in E_l\cap E_r$ and $(X\overset{\pi_X}{\to}\mathrm{pt})\in E_l\cap E_r$, then $X\in\mathrm{Corr}(\mathcal{C})_{E_l,E_r}$ is self-dual, namely the morphisms $X\times X\xleftarrow{\Delta_X} X\xrightarrow{\pi_X}\mathrm{pt}$ and $\mathrm{pt}\xleftarrow{\pi_X} X\xrightarrow{\Delta_X} X\times X$ exhibit $X\simeq X^\vee$. Moreover, if $\boxtimes: D(X)\otimes D(X)\xrightarrow{\sim} D(X\times X)$, then $D(X)$ is self-dual, with counit and unit given by the functors

$$D(X)\otimes D(X)\xrightarrow{\boxtimes} D(X\times X)\xrightarrow{\Delta_X^!} D(X)\xrightarrow{(\pi_X)_*} D(pt)$$

and

$$D(\mathrm{pt})\xrightarrow{\pi_X^!} D(X)\xrightarrow{(\Delta_X)_*} D(X\times X)\xrightarrow{\boxtimes^{\mathrm{R}}} D(X)\otimes D(X),$$

where $\boxtimes^{\mathrm{R}}$ is the right adjoint of $\boxtimes$. In particular, in this case $D(X)$ is a closed symmetric monoidal category.

(4) Given a 3-functor formalism $D:\mathrm{Corr}(\mathcal{C})_{E_l,E_r}\to\mathrm{Lincat}_E$ such that $E_l^L=E_l$ and $E_r^{\mathrm{L}}=E_r$ are left-adjointable, then the assignment $D:\mathrm{Corr}(\mathcal{C})_{E_l,E_r}\to\mathrm{Lincat}_E$ sending $X\xleftarrow{f} Z\xrightarrow{g} Y$ to $D(X)\xrightarrow{g_\dagger\circ f^\star} D(Y)$ is a *6-functor formalism* in the sense of [34, Definition 2.5].

Definition 1.5. Let $D:\mathrm{Corr}(\mathcal{C})_{E_l,E_r}\to\mathrm{Lincat}_E^?$ be a 3-functor formalism. A class of weakly stable morphisms $E_l^{\mathrm{R}}\subset E_l$ is said to be *right-adjointable* if for every $f:X\to Y\in E_l^{\mathrm{R}}$ the functor $f^!$ admits a right adjoint[4] $f_\star$. Moreover, we say that E_l^{R}

(i) is *compatible with* $E_l^{\mathrm{BC}}\subset E_l$ *base change* if $(g_Y)^!\circ f_\star\xrightarrow{\sim}(f')_\star\circ(g_X)^!$ for every $g_Y\in E_l^{\mathrm{BC}}$;

(ii) is *compatible with* $E_r^{\mathrm{PC}}\subset E_r$ *pushforward* if $(g_Y)_*\circ(f')_\star\xrightarrow{\sim} f_\star\circ(g_X)_*$ for every $g_Y\in E_r^{\mathrm{PC}}$.

(iii) *satisfies projection formula* if $f_\star(\mathscr{F})\otimes\mathscr{G}\xrightarrow{\sim} f_\star(\mathscr{F}\otimes f^!(\mathscr{G}))$ for every $\mathscr{F}\in D(X)$ and $\mathscr{G}\in D(Y)$.

For a 3-functor formalism with values in $\mathrm{Lincat}_E^{\mathrm{perf}}$ we don't require the right adjoint to be continuous. Whereas for a 3-functor formalism with values in $\mathrm{Lincat}_E^{\mathrm{c.g.}}$ we have that $f_\star$

[3] Here the morphism is induced by the co-unit of $(f_*, f^!)$, but we also have an equivalence in the other direction induced by the co-unit of $(f_\dagger, f^!)$.

[4] In the category where the sheaf theory takes values. In particular, we require that the right adjoint is continuous if $?=\emptyset$ or *c.g.*.

is continuous if and only if $f^!$ preserves compact objects. We make also formulate what it means for a weakly stable class $E_r^{\mathrm{R}} \subset E_r$ to be right-adjointable, but we won't need that.

Remark 1.6. (1) As in Remark 1.4 (1) we say that a morphism $(X \xrightarrow{f} Y) \in E_r^{\mathrm{R}} \subset E_r$ is *internally right adjointable* if $(f_*, f^!)$ is an adjunction. Similarly, $E_l^{\mathrm{R}} \subset E_l$ is *internally right adjointable* if $(f^!, f_*)$ is an adjunction.

(2) In practice, when constructing a 3-functor formalism, the functors $f_* : D(X) \to D(Y)$ and $f^! : D(Y) \to D(X)$ assigned to a morphism $(X \xrightarrow{f} Y)$ are not completely independent. One often starts from a functor $f^! : D(Y) \to D(X)$ defined for all $(X \xrightarrow{f} Y) \in E_l$ and check that it is left adjointable for a certain class E_l^{L} and right adjointable for a certain class E_l^{R}. That these adjoints can be combined to give a well-defined pushforward is the content of many sheaf extension results, see for example [20, §8.2.5] or [31, Proposition A.5.10].

1.4. **Sheaves on placid schemes.** This section reviews how to construct from the literature a sheaf formalism of ind-constructible ℓ-adic sheaves on qc qs schemes. At this level of generality this sheaf theory is only a 3-functor formalism, but it allows us to define the notion of cohomologically smoothness. Using cohomological smoothness we cut a subcategory of qcqs schemes, namely placid schemes and check that the restriction of the sheaf theory to this category yields a 6-functor formalism.

1.4.1. *Sheaves on qcqs algebraic spaces of finite type.* We assume that k is a finite field or an algebraically closed field. To any qcqs algebraic space X of finite type over k we associate the small stable ∞-category $D_{\mathrm{c}}(X, E)$ of ℓ-adic sheaves on X with values on Mod_E. Concretely, one takes [27, Notation 2.2.3] as a definition for $E = \mathbb{F}_\ell$ and use [26, §0.1] to extend it to $E = \mathbb{Z}_\ell$ and $E = \overline{\mathbb{Q}_\ell}$ (see also [20, §10.2.1]). The constructions in the remainder of this section (except for §1.4.3) are independent of the coefficient E, so we will often omit it from the notation.

The category of (ind-constructible) sheaves[5] on a qcqs algebraic space of finite type X is defined as $D(X) := \mathrm{Ind}(D_{\mathrm{c}}(X))$, i.e. the category obtained by formally adjoining all filtered colimits to $D_{\mathrm{c}}(X)$.

The stable ∞-categories $D_{\mathrm{c}}(X)$ and $D(X)$ have as underlying homotopy categories the classical triangulated categories of constructible and all ℓ-adic sheaves, respectively. If we were only interested in algebraic spaces of finite type the classical construction of these triangulated categories would have be enough. However, to extend extend the formalism to algebraic spaces not necessarily of finite type and to prestacks, there is no simple way of correctly defining these categories. For instace, two problems are descent for triangulated categories does not work well (see [27, §0.2] for a nice discussion of this issue) and the correct sheaves on prestacks not necesarrily of finite type are limits and colimits of categories, which also don't behave well for triangulated categories.

We emphasize that the sheaf formalism as developed always comes in two variants:

- $D_{\mathrm{c}}(X) \in \mathrm{Lincat}_E^{\mathrm{perf}}$ a small idempotent complete linear stable ∞-category of *constructible* sheaves, and

[5] Concretely speaking the objects of this ∞-category are (unbounded) complexes of sheaves (with possibly infinite-dimensional cohomology) up to homotopy, we follow the convention of referring to them simply as sheaves.

• $D(X) \in \mathrm{Lincat}_E^{\mathrm{c.g.}}$ a presentable linear stable ∞-category of *ind-constructible* sheaves[6].

For the rest of this subsection we write write $D_{-}(X)$ for either $D_{\mathrm{c}}(X)$ or $D(X)$.

Given any morphism $f : X \to Y$ between qcqs algebraic spaces of finite type one has adjunctions ([27, §6.2] and [26, §0.1]):

$$f^* : D_{-}(Y) \rightleftarrows D_{-}(X) : f_* \quad \text{and} \quad f_! : D_{-}(X) \rightleftarrows D_{-}(Y) : f^! \ , \tag{1.4}$$

and compatibility isomorphisms: $f_! \xrightarrow{\sim} f_*$ when f is proper and $f^* \xrightarrow{\sim} f^!$ when f is étale. There is also an adjunction given by the sheaf tensor and sheaf inner-hom, i.e. for every $\mathscr{F} \in D_{-}(X)$ one has an adjunction:

$$(-) \otimes \mathscr{F} : D_{-}(X) \rightleftarrows D_{-}(X) : \mathscr{H}\mathrm{om}(\mathscr{F}, -) \ . \tag{1.5}$$

The functors $(f^*, f_*), (f_!, f^!), ((-) \otimes \mathscr{F}, \mathscr{H}\mathrm{om}(\mathscr{F}, -))$ are first constructed for the constructible categories and then ind-extended. In this formulation, the preservation of constructibility holds by construction. Here are some of the compatibilities between the six functors above:

• given a pullback diagram:

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"'] & Y \end{tikzcd} \tag{1.6}$$

one has an equivalence: $(g_X)_* \circ f^! \xrightarrow{\sim} (f')^! \circ (g_Y)_*$.

• for any $X \in \mathrm{AlgSpc}_{\mathrm{ft}}$ the category $D(X)$ (resp. $D_{\mathrm{c}}(X)$) is a commutative algebra in $\mathrm{Lincat}_E^{\mathrm{c.g.}}$ (resp. $\mathrm{Lincat}_E^{\mathrm{perf}}$). Any morphism $f : X \to Y$ induces a map of $D_{-}(Y)$-modules $f_* : D_{-}(X) \to D_{-}(Y)$. Concretely, this gives the projection formula:

$$f_*(\mathscr{F} \otimes f^!(\mathscr{G})) \xrightarrow{\sim} f_*(\mathscr{F}) \otimes \mathscr{G} \tag{1.7}$$

for every $\mathscr{F} \in D_{-}(X)$ and $\mathscr{G} \in D_{-}(Y)$. In particular, when applied to the unit of the adjunction (f^*, f_*) one obtains:

$$f^*(\mathscr{F} \otimes \mathscr{G}) \longrightarrow f^*(\mathscr{F}) \otimes f^!(\mathscr{G}). \tag{1.8}$$

The data of the functors (1.4) and (1.5) satisfying base change and projection formulas determines 6-functor formalisms:

$$D_{\mathrm{c}} : \mathrm{Corr}(\mathrm{AlgSpc}_{\mathrm{ft}}) \longrightarrow \mathrm{Lincat}_E^{\mathrm{perf}} \quad \text{and} \quad D : \mathrm{Corr}(\mathrm{AlgSpc}_{\mathrm{ft}}) \longrightarrow \mathrm{Lincat}_E^{\mathrm{c.g.}} \tag{1.9}$$

that sends $X \xleftarrow{f} Z \xrightarrow{g} Y$ to $g_* \circ f^! : D_{-}(X) \to D_{-}(Y)$.

1.4.2. *Sheaves on qc qs algebraic spaces.* In this subsection, we extend (1.9) to qc qs algebraic spaces. This is inspired by [7, §5] and [33, §3 and §6], but we follow the construction of [20, §10].

Let $D_{\mathrm{c}} : \mathrm{AlgSpc}_{\mathrm{ft}}^{\mathrm{op}} \to \mathrm{Lincat}_E$ denote the restriction of (1.9), to the subcategory of $\mathrm{Corr}(\mathrm{AlgSpc}_{\mathrm{ft}})$ generated by the correspondences $X \longleftarrow Y \xrightarrow{\mathrm{id}_Y} Y$, where $X, Y \in \mathrm{AlgSpc}_{\mathrm{ft}}$. We consider:

$$\mathrm{LKE}_{\mathrm{AlgSpc}_{\mathrm{ft}} \hookrightarrow \mathrm{AlgSpc}}(D_{\mathrm{c}}) : \mathrm{AlgSpc}^{\mathrm{op}} \longrightarrow \mathrm{Lincat}_E^{\mathrm{perf}} \tag{1.10}$$

[6] Though, in some cases these will not be compactly generated.

the left Kan extension via the inclusion of schemes of finite type into all qc qs schemes.

Consider $\mathrm{Corr}(\mathrm{AlgSpc})_{all;fp}$ the category of correspondences whose objects are qcqs algebraic spaces and morphism are correspondences $X \xleftarrow{f} Z \xrightarrow{g} Y$, where g is finitely presented. By [19, Theorem 10.15] one has an extension of (1.10) to a 3-functor formalism:

$$D_{\mathrm{c}} : \mathrm{Corr}(\mathrm{AlgSpc})_{all;fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{perf}}. \tag{1.11}$$

We then let

$$D : \mathrm{Corr}(\mathrm{AlgSpc})_{all;fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{c.g.}} \tag{1.12}$$

be the Ind-extension of (1.11).

We stress that the theory (1.12) is different than directly defining étale sheaves on qcqs schemes following §1.4.1. Indeed, for $X = \operatorname{Spec} K$ where K is a field over $\mathbf{k}$ the category defined in this section for finite coefficients recovers representations of $\mathrm{Gal}(K/\mathbf{k})$ which are filtered colimits of finitely generated smooth representations, whereas the later option would consider all representations. We refer to [20, Example 10.23] for details.

1.4.3. *Cohomological smoothness.* In this section we need to pass to a dual sheaf theory. Let

$$D^*(-, \mathbb{F}_\ell) : \mathrm{Corr}(\mathrm{AlgSpc})_{all;fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{c.g.}} \tag{1.13}$$

denote the sheaf theory obtained by composing (1.12) with the duality $\mathrm{Lincat}_E^{\mathrm{c.g.}} \xrightarrow{\sim} \mathrm{Lincat}_E^{\mathrm{c.g.}}$. By §1.1 (i) this is concretely given by $D^*(X, \mathbb{F}_\ell) \simeq \mathrm{Ind}(D_{\mathrm{c}}(X, \mathbb{F}_\ell)^{\mathrm{op}})$. The functor (1.13) sends a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ to the conjugates functors (see [20, §7.2.2] for a definition) $(f^!)^\circ : D^*(X, \mathbb{F}_\ell) \to D^*(Z, \mathbb{F}_\ell)$ and $(g_*)^\circ : D^*(Z, \mathbb{F}_\ell) \to D^*(Y, \mathbb{F}_\ell)$ of $f^!$ and g_*, respectively.

Let $((f^!)^\circ, f_\dagger)$ and $((g_*)^\circ, g^\star)$ denote the adjunctions, where g is finitely presented. The restriction of (1.12) to algebraic spaces of finite type recovers the usual 6-functor formalism. Verdier duality allows us to identify these pair of adjunctions with (f^*, f_*) and $(g_!, g^!)$, since $(f^!)^\circ$ is the Ind-extension of f^* (see [20, §10.4.1] for more details). For the rest of this subsection we use the later notation; but we warn the reader that this is different than the notation outside this subsection.

Let $f : X \to Y$ be a finitely presented morphism and consider the diagram:

$$\begin{tikzcd} X \underset{Y}{\times} X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

the unit of the adjunction (f^*, f_*) gives a map $f^!(\omega_Y) \to f^! \circ f_* \circ f^!(\omega_Y)$, by base change and using the adjunction $(p_2^*, p_{2,*})$ we obtain:

$$p_2^* \circ f^!(\omega_Y) \longrightarrow p_1^! \circ f^*(\omega_Y). \tag{1.14}$$

Notice that here the functors f^* and p_2^* exist without any further assumption because we are considering the sheaf theories obtained by applying $D^*(-, \mathbb{F}_\ell)$, but they *a priori* do not preserve compact objects.

Definition 1.7. A morphism $f : X \to Y$ in AlgSpc is *cohomologically smooth* if it satisfies:

(i) f is *ULA*, i.e. f is fp and the canonical map (1.14) is an isomorphism.

(ii) $f^*(\omega_Y)$ is an invertible object of $D^*(X, \mathbb{F}_\ell)$.

Remark 1.8. (1) Any étale morphism $f : X \to Y$ is cohomologically smooth since $f^* \xrightarrow{\sim} f^!$.

(2) For any $n \geq 0$ and projection $f : \mathbf{A}^n_Y \to Y$ we have $f^*\omega_Y \xrightarrow{\sim} \omega_{\mathbf{A}^n_Y} \langle n \rangle$.

(3) By [35, Lemma 054L] (1) and (2) imply that any smooth morphism is cohomologically smooth and similarly that any perfectly smooth morphisms (see [19, Definition 10.4] for this notion) between perfect algebraic spaces is also cohomologically smooth.

(4) Given any field K over k, $f : X \to \operatorname{Spec} K$ is cohomologically smooth if and only if $f^!\overline{\mathbb{Q}}_{\ell,\operatorname{Spec} K}\big|_{X_i} \simeq \overline{\mathbb{Q}}_{\ell,X_i} \langle d_i \rangle$, where $d_i = \dim X_i$ and X_i is an irreducible component of X. One can refer to this condition as *rationally smooth* (cf. [8, §1.1 Definition] in the case where $\operatorname{char} k = 0$). This also shows how cohomologically smooth is strictly more general than smoothness.

(5) For $f : X \to Y$ cohomologically smooth one has

$$\omega_X \langle d_f \rangle \xrightarrow{\sim} f^*\omega_Y, \tag{1.15}$$

where d_f is the relative cohomological dimension of f (see [20, §10.3.3] for a precise definition) and $\langle d_f \rangle$ represents the cohomological and Tate twist by d_f. This result is a bit subtle to prove and follows a density argument, see [20, Proposition 10.45].

(6) By [20, Remark 8.33], the heuristic for Defintion 1.7 is that the notion of dualizability of $D^*(X, \mathbb{F}_\ell)$ as a $D^*(\mathrm{pt}, \mathbb{F}_\ell)$-module category is too strict, this is equivalent to requiring that X is finitely presented and that the exterior tensor product of the sheaf theory is an equivalence, which does not hold in general. Whereas the condition that X is dualizable in $\mathrm{Corr}(\mathrm{AlgSpc})_{all;fp}$ simply imposes that X is finitely presented which is too weak. The correct notion turns out to be to require that (X, ω_X) is dualizable in the category of cohomological correspondences as defined in [34, Lecture V].

1.4.4. *Placid algebraic spaces.* Let $f : X \to Y$ be a morphism in AlgSpc we say that:

(a) f is *cohomologically pro-smooth* if there is a presentation $X \xrightarrow{\sim} \lim_I X_i$ as a cofiltered limit where: each $X_i \to X_j$ is cohomologically smooth affine and each $X_i \to Y$ is cohomologically smooth;

(b) f is *strongly cohomologically pro-smooth* if in addition to ((a)) the structure morphisms $X_i \to X_j$ are surjective;

(c) f is *weakly cohomologically pro-smooth* if there is a surjective cohomologically pro-smooth map $U \to X$, such that the composite $U \to X \to Y$ is cohomologically pro-smooth;

(d) f is *essentially cohomologically pro-smooth* if there is a factorization $X \to X' \to Y$ where $X \to X'$ is cohomologically pro-smooth and $X' \to Y$ is finitely presented.

A qcqs algebraic space $X \in \mathrm{AlgSpc}$ is said to be *placid* if the morphism $X \to \operatorname{Spec} k$ is essentially cohomologically pro-smooth. We let $\mathrm{AlgSpc}_{\mathrm{pl}} \hookrightarrow \mathrm{AlgSpc}$ denote the subcategory of placid schemes.

More generally, given $f : \mathscr{X} \to \mathscr{Y}$ a morphism in PStk we say that:

(d) f is (resp.*strongly*, *essentially*) *cohomologically pro-smooth* if for every affine scheme $S \to \mathscr{Y}$ the fiber product $S \underset{\mathscr{Y}}{\times} \mathscr{X}$ is a qcqs algebraic space and the induced morphism $S \underset{\mathscr{Y}}{\times} \mathscr{X} \to S$ is (resp. strongly, essentially) cohomologically pro-smooth;

(e) f is *weakly cohomologically pro-smooth* if for every affine scheme $S \to \mathscr{Y}$ there is a jointly surjective family $\{T_i \to S \underset{\mathscr{Y}}{\times} \mathscr{X}\}_I$ with $T_i \in \mathrm{AlgSpc}$ where each composite $T_i \to S \underset{\mathscr{Y}}{\times} \mathscr{X} \to S$ is cohomologically pro-smooth.

1.4.5. *Sheaves on placid algebraic spaces.* Let $\mathrm{AlgSpc}_{\mathrm{pl}} \hookrightarrow \mathrm{AlgSpc}$ denote the subcategory of placid algebraic spaces. We have:

Proposition 1.9. *[20, Proposition 10.69] The restriction of (1.12) to:*

$$D : \mathrm{Corr}(\mathrm{AlgSpc}_{\mathrm{pl}})_{fp;fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{c.g.}} \tag{1.16}$$

gives a 3-functor formalism on placid algebraic spaces, such that:

(i) the class $E_{et} \subset E_r$ of étale morphism is internally left adjointable and the class $E_p \subset E_l$ of fp proper morphisms is internally left adjointable;

(ii) the class $(X \xrightarrow{f} Y) \in E_{fp} \subset E_r$ of finitely presented morphisms is left adjointable with adjoint f_* compatible with weakly cohomologically pro-smooth pullbacks;

(iii) the class $(X \xrightarrow{f} Y) \in E_{fp} \subset E_l$ of finitely presented morphisms is left adjointable with adjoint $f_!$ compatible with weakly cohomologically pro-smooth pullbacks.

Moreover, the left adjoints of (ii-iv) preserve compact, i.e. constructible, objects.

Here is an heuristic of why this works, we refer the reader to [20, Proposition 10.69] for details. Consider $f : X \to Y$ a finitely presented morphism between placid algebraic spaces, then we can find $Y \xrightarrow{\sim} \lim_I Y_i$ a placid presentation of Y such that for some $i \in I$ one has a pullback diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X_i \arrow[r, "f_i"'] & Y_i \end{tikzcd},$$

where $X_i := Y_i \underset{Y}{\times} X$. Thus, to produce a left adjoit to $f_* : D(X) \to D(Y)$ it is enough to check that the diagrams:

$$\begin{tikzcd} D(X_{i+1}) \arrow[r, "(f_{i+1})_*"] & D(Y_{i+1}) \\ D(X_i) \arrow[u, "(a^X_{i+1,i})^!"] \arrow[r, "(f_i)_*"'] & D(Y_i) \arrow[u, "(a^Y_{i+1,i})^!"'] \end{tikzcd}$$

are horizontally left adjointable. Indeed, by (1.15) the canonical map:

$$(f_{i+1})^* \circ (a^Y_{i+1,i})^* \langle d_{i+1,i} \rangle \simeq (f_{i+1})^* \circ (a^Y_{i+1,i})^! \longrightarrow (a^X_{i+1,i})^! \circ (f_i)^* \simeq (a^X_{i+1,i})^* \langle d_{i+1,i} \rangle \circ (f_i)^*$$

is an isomorphism, since $a^X_{i+1,i}$ and $a^Y_{i+1,i}$ are cohomologically smooth of same relative dimension. A similar argument proves that $f^! : D(Y) \to D(X)$ has a left adjoint for f finitely presented.

The arguments for base change are a bit more involved but follow the same logic. This is proved in the context of sheaves on perfect qcqs algebraic spaces in [20, Proposition 10.69] by constructing a 6-functor $*$-theory and passing to the dual categories.

1.4.6. *Variant: Sheaves on perfect qcqs algebraic spaces.* The sheaf formalisms of this section will equally apply to the context of perfect geometry. More precisely, by restricting (1.12) to the subcategory $\mathrm{AlgSpc}^{\mathrm{perf}}$ via (1.2) we obtain a 3-functor formalism. Further restricting it to the subcategory $\mathrm{AlgSpc}^{\mathrm{perf}}_{\mathrm{pl}}$ of placid perfect algebraic spaces we obtain a 3-functor formalism

$$D : \mathrm{Corr}(\mathrm{AlgSpc}^{\mathrm{perf}}_{\mathrm{pl}})_{pfp;pfp} \longrightarrow \mathrm{Lincat}^{\mathrm{c.g.}}_E \tag{1.17}$$

that satisfies the same conditions as in Proposition 1.9 but with perfectly finitely presented morphisms in place of finitely presented morphisms.

Notation 1.10. In the remaining subsections of this section the extensions of the sheaf theory will come in two versions: one for non-perfect objects and the other for perfect objects; the only difference is that for perfect objects one should always consider perfectly finitely presented instead of the more strict finitely presented. All the arguments are formally the same except for this difference. Thus, we will use PStk, AlgSpc, $\mathrm{AlgSpc}_{\mathrm{pl}}$ and any other further decoration introduced below to either mean the non-perfect or perfect version of these objects, for instance *fp* should be read perfectly finitely presented or finitely presented depending on the context.

1.5. **Sheaves on sifted-placid stacks.** In extending the sheaf theory to prestacks or a subcategory of nice geometry objects, one encounters the problem that compact objects are scarse for the naive definition. In other words, the small and large categories versions of sheaf theory encoded in (1.11) and (1.12) are not compatible when extended to prestacks. This is similar to the problem with ind-coherent sheaves vs quasi-coherent sheaves, except here the more natural object is ind-coherent sheaves as opposed to quasi-coherent.

1.5.1. *Sheaves on prestacks.* Before extending the sheaf theory to the objects we are interested in this article, we need a 3-functor formalism on prestacks to formulate certain conditions.

We define a sheaf theory on prestacks:

$$D : \mathrm{Corr}(\mathrm{PStk})_{all;fp} \longrightarrow \mathrm{Lincat}_E \tag{1.18}$$

by taking the right Kan extension of the functor (1.12) via the inclusion $\mathrm{Corr}(\mathrm{AlgSpc})_{all;fp} \hookrightarrow \mathrm{Corr}(\mathrm{PStk})_{all;fp}$.

In fact, by [20, Theorem 10.91] we have an further extension of (1.18):

$$D : \mathrm{Corr}(\mathrm{PStk})_{all;ind-fp} \longrightarrow \mathrm{Lincat}_E, \tag{1.19}$$

where we allow $*$-pushforward for any ind-fp morphism. This construction happens in two steps. First, one extends (1.12) to the category of ind-algebraic spaces IndAlgSch

$$D : \mathrm{Corr}(\mathrm{IndAlgSch})_{all;ind-fp} \longrightarrow \mathrm{Lincat}_E, \tag{1.20}$$

using an argument similar to that in §1.5.4. Second, one takes the right Kan extension of (1.20) via the inclusion $\mathrm{Corr}(\mathrm{IndAlgSch})_{all;ind-fp} \hookrightarrow \mathrm{Corr}(\mathrm{PStk})_{all;ind-fp}$ using [10, Chapter 8, Theorem 6.1.5] or [19, Proposition 8.43].

For the 3-functor formalism (1.19) has representable étale morphism as internally left adjointable in E_r, i.e. $(f^!, f_*)$ is an adjuncion, and representable fp proper morphisms as internally left adjointable morphisms in E_l, i.e. $(f_*, f^!)$ is adjunction in this case. In particular, we obtain that given $\imath : \mathscr{Z} \hookrightarrow \mathscr{X}$ a fp closed embedding and $\jmath : \mathscr{U} \hookrightarrow \mathscr{X}$ is qcqs open complement one has a half-recollement:

$$\begin{tikzcd} D(\mathscr{U}) \arrow[r, leftarrow, "\jmath^!"] \arrow[r, bend right, "\jmath_*"'] & D(\mathscr{X}) \arrow[r, leftarrow, "\imath_*"] \arrow[r, bend right, "\imath^!"'] & D(\mathscr{Z}). \end{tikzcd}$$

This is enough to glue individual sheaves, since for any $\mathscr{F} \in D(\mathscr{X})$ we obtain:

$$\imath_* \circ \imath^!(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \jmath_* \circ \jmath^!(\mathscr{F}).$$

However, we can't glue the whole category $D(\mathscr{X})$ from $D(\mathscr{U})$ and $D(\mathscr{Z})$ since this requires a left adjoint to $\imath_*$. In the next subsection we will remind this by restricting the sheaf theory to well-behaved colimits of stacks such that we have this further left adjoint to finitely presented inclusions.

1.5.2. *Sifted-placid stacks.* Before proceeding with the construction of the sheaf formalism we need to introduce the type of geometric objects on which this sheaf theory will behave nicely. The following is a summary of [20, §10.5.1 and §10.6.1] to which we refer the reader for more details and slightly more general notions that might be useful in future directions.

Definition 1.11. Let $\mathscr{X} \in \mathrm{PStk}$ be an étale sheaf. We will say that $\mathscr{X}$ is:

(a) a *placid stack* if there exist a *placid atlas* $h_{\mathscr{X}} : U \to \mathscr{X}$, i.e. a placid algebraic space U and a representable strongly cohomology smooth morphism $h_{\mathscr{X}}$ such that

$$D(\mathscr{X}) \xrightarrow{\sim} \lim D((U/\mathscr{X})^\bullet), \tag{1.21}$$

where $(U/\mathscr{X})^\bullet$ is the Cech nerve of $h_{\mathscr{X}}$.

(b) an *ind-placid stack* if there exists a presentation $\mathrm{colim}_I\, \mathscr{X}_i \xrightarrow{\sim} \mathscr{X}$, where each $\mathscr{X}_i$ is a placid stack and $\mathscr{X}_i \hookrightarrow \mathscr{X}_j$ are finitely presented morphisms.

(c) a *sifted-placid stack* if it admits a *sifted-placid atlas*, i.e. there exist $\mathscr{Y}$ an ind-placid stack and $h_{\mathscr{X}} : \mathscr{Y} \to \mathscr{X}$ a surjective ind-fp proper morphism.

Remark 1.12. Given a geometric context, the usual definition of a stack would only require that $U \to \mathscr{X}$ is a cover for the étale (or fpqc) topology. However, the sheaf theory we consider in this article does not have descent in this generality, so we impose (1.21) in the definition. We notice that if $f : U \to \mathscr{X}$ is ind-fp proper or representable essentially cohomologically pro-smooth, then (1.21) automatically holds (see [20, Proposition 10.99 and Proposition 10.101]).

Remark 1.13. (1) Any Artin stack of finite presentation (or its perfection) is a placid stack over k.

(2) Let $H \xrightarrow{\sim} \lim_I H_i$ be affine group scheme, where each H_i is a reduced (perfectly) finite type group scheme and $H_i \to H_j$ are (perfectly) smooth affine morphisms. Given X a placid scheme, then the étale quotient stack X/H is a placid stack.

(3) Given $\mathscr{X} \to \mathscr{Z}$ an ind-fp (resp. proper) morphism of étale stacks such that $\mathscr{Z}$ is a placid stack, then $\mathscr{X}$ is an ind-placid stack with a presentaiton $\mathrm{colim}_I\, \mathscr{X}_i \xrightarrow{\sim} \mathscr{X}$ such that $\mathscr{X}_i \to \mathscr{X} \to \mathscr{Z}$ is fp (resp. proper). This result is [20, Lemma 10.142] and is crucial to prove certain results for the sheaf theory on sifted-placid stacks below.

(4) Given $\mathscr{Y}$ an ind-placid stack and $\mathscr{H}$ an ind-placid group stack which acts on $\mathscr{Y}$ via ind-fp proper morphisms, then $\mathscr{X} \simeq \mathscr{Y}/\mathscr{H}$ is a sifted-placid stack.

1.5.3. *Ind-finite sheaves on placid stacks.* The starting point for a good theory of sheaves on placid stacks is the constructible 3-functor formalism from (1.11). Let $\mathrm{Stk}_{\mathrm{pl}}$ denote the category of placid stacks, we have an extension:

$$D_{\mathrm{c}} : \mathrm{Corr}(\mathrm{Stk}_{\mathrm{pl}})_{all;fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{perf}} \tag{1.22}$$

given by right Kan extension via $\mathrm{Corr}(\mathrm{AlgSpc})_{all;fp} \hookrightarrow \mathrm{Corr}(\mathrm{Stk}_{\mathrm{pl}})_{all;fp}$. We take the Ind-completion of (1.22) to define:

$$D_{\mathrm{ind-fin.}} : \mathrm{Corr}(\mathrm{Stk}_{\mathrm{pl}})_{all;fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{c.g.}}. \tag{1.23}$$

This sheaf theory is refered to as renormalized in [2, Appendix F.5] and [21]. We will refer to it as ind-finite.

Notice that by construction we have

$$D_{\mathrm{c}}(\mathscr{X}) \xrightarrow{\sim} \lim D_{\mathrm{c}}((U/\mathscr{X})^{\bullet}), \tag{1.24}$$

where $U \to \mathscr{X}$ is a placid atlas. We have

Proposition 1.14. *Let $E_{w.c.p.s.} \subset E_l$ denote the class of weakly cohomologically pro-smooth morphisms. The 3-functor formalism of (1.23) satisfies:*

- *(i) the class $E_{et} \subset E_r$ of representable étale morphisms is internally left adjointable and the class $E_p \subset E_l$ of representable fp proper morphism is internally left adjointable;*
- *(ii) the class $E_{fp} \subset E_l$ of representable fp morphisms is left adjointable with $E_{w.c.p.s.}$ compatible base change;*
- *(iii) the class $E_{fp} \subset E_r$ of finitely presented morphisms is left adjointable with $E_{w.c.p.s.}$ compatible base change.*

The proposition is proved by reducing using descent to reduce to Proposition 1.9. See [20, Proposition 10.114] for details.

Since the categories $D(\mathscr{X})$ have all colimits one has a natural functor:

$$\Psi : D_{\mathrm{ind-fin.}}(\mathscr{X}) \longrightarrow D(\mathscr{X})$$

but this is far from an equivalence. When $\mathscr{X}$ is a placid stack with a nice cover, then Ψ is an equivalence after left completion (see [19, Lemma 10.123] for the precise statement).

1.5.4. *Sheaves on ind-placid stacks.* Let $\mathrm{IndStk}_{\mathrm{pl}}$ denote the category of ind-placid stacks. We extend the theory (1.22) to:

$$D_{\mathrm{c}} : \mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}})_{all;fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{perf}} \tag{1.25}$$

by taking its right Kan extension via the inclusion $\mathrm{Corr}(\mathrm{Stk}_{\mathrm{pl}})_{all;fp} \hookrightarrow \mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}})_{all;fp}$. Notice that for an ind-placid stack $\mathscr{X}$ given a presentation $\mathrm{colim}_I\, \mathscr{X}_i \xrightarrow{\sim} \mathscr{X}$ one has:

$$\operatorname*{colim}_I D_{\mathrm{c}}(\mathscr{X}_i) \xrightarrow{\sim} D_{\mathrm{c}}(\mathscr{X}).$$

We then consider

$$D_{\mathrm{ind-fin.}} : \mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}})_{all;fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{c.g.}} \tag{1.26}$$

by taking the Ind-extension of (1.25). The functor (1.26) admits an extension to:

$$D_{\mathrm{ind-fin.}} : \mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}})_{all;ind-fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{c.g.}}, \tag{1.27}$$

whose restriction to $\mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}})_{fp;fp}$ satisfies an analogue of Proposition 1.14 (see [20, Theorem 10.150 (2)]).

The extension (1.27) is constructed in two steps (see [10, Chapter 8, Theorem 1.1.9] or [19, Corollary 8.48]). The first extension D_1 is determined by the following diagram:

$$\begin{tikzcd}
\mathrm{Stk}_{\mathrm{pl}}^{\mathrm{op}} \arrow[r, "\imath_1"] \arrow[d, hook] & \mathrm{IndStk}_{\mathrm{pl}}^{\mathrm{op}} \arrow[d, hook, "\jmath"] \\
\mathrm{Corr}(\mathrm{Stk}_{\mathrm{pl}})_{all;fp} \arrow[r] \arrow[d, "D_{\mathrm{ind-fin.}}"'] & \mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}})_{all;fp} \arrow[dl, "D_1"] \\
\mathrm{Lincat}_E^{\mathrm{c.g.}} &
\end{tikzcd}$$

and the condition that $D_1 \circ \jmath \xrightarrow{\sim} \mathrm{RKE}_{\imath_1}(D_{\mathrm{ind-fin.}})$. The second is determined by:

$$\begin{tikzcd}
\mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}})_{all;fp} \arrow[r, "\imath_2"] \arrow[d, "D_1"'] & \mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}})_{all;ind-fp} \arrow[dl, "D_2"] \\
\mathrm{Lincat}_E^{\mathrm{c.g.}} &
\end{tikzcd}$$

where D_2 is the operadic left Kan extension of D_1 via $\imath_2$. Heuristically, this uses that for any ind-placid stack $\mathscr{X}$ one has $\mathrm{colim}_{\mathrm{J}_{\mathscr{X}}} D(\mathscr{Y}) \xrightarrow{\sim} D(\mathscr{X})$ where

$$\mathrm{J}_{\mathscr{X}} = \{\mathscr{Y} \xrightarrow{f} \mathscr{X} \mid \mathscr{Y} \text{ placid stack } f \text{ fp-closed embedding}\},$$

to extend the sheaf theory.

1.5.5. *Sheaves on sifted-placid stacks.* Let $\mathrm{sIndStk}_{\mathrm{pl}}$ denote the category of sifted-placid stacks. We notice that Remark 1.13 (3) implies that $\imath : \mathrm{Corr}(\mathrm{IndStk}_{\mathrm{pl}}) \hookrightarrow \mathrm{Corr}(\mathrm{sIndStk}_{\mathrm{pl}})$ is fully faithful.

Proposition 1.15. *The left Kan extension of (1.27) via $\imath$, gives a 3-functor formalism*

$$D_{\mathrm{ind-fin.}} : \mathrm{Corr}(\mathrm{sIndStk}_{\mathrm{pl}})_{all;ind-fp} \longrightarrow \mathrm{Lincat}_E^{\mathrm{c.g.}} \tag{1.28}$$

satisfying:

(i) *the class $E_{et} \subset E_r$ of representable étale morphisms is internally left adjointable and the class $E_{ind-f.p.p.} \subset E_r$ of representable ind-finitely presented proper morphisms is internally left adjointable;*

(ii) *the class $(\mathscr{X} \xrightarrow{f} \mathscr{Y}) \in E_{ind-f.p} \subset E_l$ of representable ind-fp morphisms is left adjoitable with adjoint $f_!$ which is compatible with weakly cohomologically pro-smooth base change;*

(iii) *the class $(\mathscr{X} \xrightarrow{f} \mathscr{Y}) \in E_{f.p.} \subset E_r$ of representable finitely presented morphism is left adjointable with adjoint f^* which is compatible with weakly cohomologically pro-smooth morphism pullbacks.*

Remark 1.16. For $\mathscr{X}$ a sifted-placid stack the following hold:

(1) one has an equivalence $\mathrm{colim}_{\mathrm{I}_{\mathscr{X}}} D_{\mathrm{ind-fin.}}(\mathscr{Y}) \xrightarrow{\sim} D_{\mathrm{ind-fin.}}(\mathscr{X})$, where

$$\mathrm{I}_{\mathscr{X}} = \{\mathscr{Y} \xrightarrow{f} \mathscr{X} \mid \mathscr{Y} \text{ placid stack } f \text{ ind-fp}\}. \tag{1.29}$$

(2) the category $D_{\mathrm{ind-fin.}}(\mathscr{X})$ is compactly generated and $D_{\mathrm{ind-fin.}}(\mathscr{X})^{\omega}$ is generated by objects of the form $h_{\mathscr{Y},*}(\mathscr{F}_{\mathscr{Y}})$, where $g_{\mathscr{Y}} : \mathscr{Y} \to \mathscr{X}$ is ind-fp, $\mathscr{Y}$ is a placid stack and $\mathscr{F}_{\mathscr{Y}} \in D_{\mathrm{c}}(\mathscr{Y})$.

For further reference we summarize what concretely is encoded in the functor (1.28).

Lemma 1.17. *Any correspondence $\mathscr{X} \xleftarrow{f} \mathscr{Z} \xrightarrow{g} \mathscr{Y}$ of sifted-placid stacks, where g is ind-fp morphism is sent to $g_* \circ f^! : D_{\mathrm{ind-fin.}}(\mathscr{X}) \to D_{\mathrm{ind-fin.}}(\mathscr{Y})$.*

(i) For any diagram

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "g_{\mathscr{X}}"] \arrow[d, "f'"'] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{Y}' \arrow[r, "g_{\mathscr{Y}}"'] & \mathscr{Y} \end{tikzcd}, \tag{1.30}$$

where $g_{\mathscr{Y}}$ is ind-fp, one has an equivalence: $(g_{\mathscr{X}})_* \circ (f')^! \simeq f^! \circ (g_{\mathscr{X}})_*$;

(ii) If f is representable ind-fp proper then f_* is left adjoint to $f^!$.

(iii) If f is representable ind-fp one has an adjunction $(f_!, f^!)$ such that $g^!_{\mathscr{Y}} \circ f_! \xrightarrow{\sim} f'_! \circ g^!_{\mathscr{X}}$.

(iv) If g is representable fp-morphism then one has an adjunction (g^*, g_*) such that for any pullback diagram

$$\begin{tikzcd} \mathscr{Z}' \arrow[r, "h_{\mathscr{Z}}"] \arrow[d, "g'"'] & \mathscr{Z} \arrow[d, "g"] \\ \mathscr{Y}' \arrow[r, "h_{\mathscr{Y}}"'] & \mathscr{Y} \end{tikzcd},$$

where $h_{\mathscr{Y}}$ is weakly cohomologically pro-smooth, one has $(g')^* \circ h^!_{\mathscr{Y}} \xrightarrow{\sim} h^!_{\mathscr{Z}} \circ g^*$.

(v) If f is representable ind-fp-proper and $g_{\mathscr{Y}}$ is representable fp then $f_! \xrightarrow{\sim} f_*$ and we have $f'_* \circ g^*_{\mathscr{X}} \xrightarrow{\sim}{}^* g_{\mathscr{Y}} \circ f_*$ obtained by passing to left adjoints on (i).

We also spell out the open-closed gluing properties that the sheaf theory (1.28) satisfy.

Lemma 1.18. *[19, Proposition 10.161] Let $\imath : \mathscr{Z} \hookrightarrow \mathscr{X}$ be a closed embedding of sifted-placid stacks with complement $\jmath : \mathscr{U} \hookrightarrow \mathscr{X}$. If $\imath$ is finitely presented, equivalently $\jmath$ is qcqs, then we have a recollement diagram:*

$$\begin{tikzcd} D_{\mathrm{ind-fin.}}(\mathscr{U}) \arrow[r, bend left, "\jmath_!"] \arrow[r, bend right, "\jmath_*"'] & D_{\mathrm{ind-fin.}}(\mathscr{X}) \arrow[l, "\jmath^!"'] \arrow[r, bend left, "\imath^*"] \arrow[r, bend right, "\imath^!"'] & D_{\mathrm{ind-fin.}}(\mathscr{Z}) \arrow[l, "\imath_*"'] \end{tikzcd}.$$

In particular, for every $\mathscr{F} \in D(\mathscr{X})$ we have cofiber-fiber sequences:

$$\imath_*\imath^!\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \jmath_*\jmath^!\mathscr{F} \qquad \jmath_!\jmath^!\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \imath_*\imath^*\mathscr{F}. \tag{1.31}$$

Notation 1.19. In the rest of this article we will simply write $D(\mathscr{X})$ for $D_{\mathrm{ind-fin.}}(\mathscr{X})$. This should not cause confusion since we will never use the categories $D(\mathscr{X})$ in what follows.

1.6. **Renormalized pushforward.** For the computation of the co-center of the affine Hecke category in §4 we will need continuous right adjoints to !-pullbacks, with sufficient base change. These will exist in a rather large generality for our sheaf theory. In this section we explain the ingredients necessary to make sense of that.

1.6.1. *Representable pushforward.* Let $f : X \to Y$ be a cohomologically smooth morphism between qcqs algebraic spaces, then by (1.15) we have:

(i) $f^*(\omega_Y)$ is constructible;

(ii) the natural transformation $f^* \xrightarrow{\sim} f^*(\omega_Y) \otimes f^!$, induced by taking $\mathscr{F} = \omega_X$ in (1.8), is an isomorphism, so $f^!$ preserves constructible sheaves.

In particular, the functor $f_*^{\text{ren}}(\mathscr{F}) := f_*(f^*(\omega_Y) \otimes \mathscr{F})$ is a continuous right adjoint to $f^!$.

We can extend the renormalized pushforward to cohomologically pro-smooth morphisms as follows:

Proposition 1.20. *[19, Proposition 10.74] Let $f : X \to Y$ be a cohomologically pro-smooth morphism between qcqs algebraic spaces. Consider a factorization $f : X = \lim_I X_i \xrightarrow{f_i} Y$ where $f_i : X_i \to Y$ are cohomologically smooth. We define:*

$$f_*^{\text{ren}} : D(X) \longrightarrow D(Y), \qquad f_*^{\text{ren}}(\mathscr{F}) := \operatorname*{colim}_I (f_i)_*(f_i^!(\mathscr{F})).$$

Then the class of cohomologically pro-smooth morphisms is right-adjointable and satisfies projection formula.

The idea behind the proof of Proposition 1.20 is to start with the result for cohomologically smooth morphisms, which holds by the projection formula for $(f_*, f^!)$. Then one extends it to cohomologically pro-smooth morphisms using the formula

$$f_*(\mathscr{F}) \simeq \operatorname*{colim}_{I,J} (a^X_{j',j})^* \circ (f_{i',j'})_* \circ (a^Y_{i',i}),$$

where $X \xrightarrow{\sim} \lim_J (X_{j'} \xrightarrow{a^X_{j',j}} X_j)$ and $Y \xrightarrow{\sim} \lim_I (Y_{i'} \xrightarrow{a^Y_{i',i}} Y_i)$ are presentations, and $f_{i',j'} : X_{i'} \to Y_{j'}$ is cohomologically smooth.

In fact, using the descent condition (1.24) we can also define a renormalized pushforward for any $f : \mathscr{X} \to \mathscr{Y}$ be a representable cohomologically pro-smooth morphism between placid stacks. Proposition 1.20 then implies that the class of representable cohomologically pro-smooth morphisms is nicely right-adjointable for the sheaf theory (1.23).

1.6.2. *Non-representable pushforward.* In fact, for the argument that computes the cocenter of the affine Hecke category we need to have a continuous right adjoints to pullbacks via morphisms whose fiber is isomorphic to $\frac{\text{pt}}{H}$, where H is a cohomologically pro-smooth group scheme. We start by explaining how to extend the construction of §1.6.1 to weakly cohomologically pro-smooth morphisms.

Another way to think of the renormalized pushforward is by considering the sheaf theory $D^* : \text{Corr}(\text{AlgSpc})_{all,fp} \to \text{Lincat}_E^{\text{c.g.}}$ from (1.13). For any $X \in \text{AlgSpc}$ the category $D^*(X)$ acts on $D(X)$. Then, one notices that:

(1) for $X \in \text{AlgSpc}_{\text{pl}}$ there is a generalized dualizing sheaf $\eta_X \in D(X)$ such that the $D^*(X)$ action induces an equivalence: $\eta_X : D^*(X) \xrightarrow{\sim} D(X)$;

(2) for $f : X \to Y$ a pro-smooth morphism between placid algebraic spaces, given an equivalence $\eta_Y : D^*(Y) \xrightarrow{\sim} D(Y)$ then $f^*(\eta_Y) : D^*(X) \xrightarrow{\sim} D(X)$ is also an equivalence.

Thus, by definition of f_*^{ren} we have a commutative diagram:

$$
\begin{tikzcd}
D(X) \arrow[r, "\eta_X^{-1}"] \arrow[d, "f_*^{\mathrm{ren}}"'] & D^*(X) \arrow[d, "f_\dagger"] \\
D(Y) & D^*(Y) \arrow[l, "\eta_Y"]
\end{tikzcd}, \tag{1.32}
$$

where $f_\dagger : D^*(X) \to D^*(Y)$ is the right adjoint to always defined the $*$-pullback $f^* : D^*(Y) \to D*(X)$, whose restriction to algebraic spaces of finite type is the usual f_*.

(3) Now given $f : X \to Y$ a *weakly* cohomologically pro-smooth morphism in the category of placid algebraic spaces. We extend the definition of renormalized pushforward by imposing that the diagram (1.32) commutes. We claim that this still satisfies Conditions (i-iii) from Proposition 1.20 (see [20, Proposition 10.75] for details).

Given $f : \mathscr{X} \to \mathscr{Y}$ a weakly cohomologically pro-smooth morphism between placid stacks, by [20, Lemma 10.111 (3)] there are placid atlases $h_{\mathscr{Y}} : U_{\mathscr{Y}} \to \mathscr{Y}$ and $h_{\mathscr{X}} : U_{\mathscr{X}} \to \mathscr{X}$ and a cohomologically pro-smooth morphism $f' : U_{\mathscr{X}} \to U_{\mathscr{Y}}$ such that $h_{\mathscr{Y}} \circ f' = f \circ h_{\mathscr{X}}$. Thus, we define:

$$f_*^{\mathrm{ren}} : D(X) \longrightarrow D(Y), \qquad f_*^{\mathrm{ren}} := (h_{\mathscr{Y}})_*^{\mathrm{ren}} \circ (f')_*^{\mathrm{ren}} \circ h_{\mathscr{X}}^!.$$

The base change that we need for this renormalized push-forward is somewhat tricky. To explain it we need to introduce a couple of concepts. A morphism $f : X \to Y$ is said to be *cohomologically unipotent* if f is cohomologically smooth and for every geometric point $\overline{y} \to Y$ the cohomology of $\overline{y} \underset{Y}{\times} X$ is acyclic.

We have the following analogues of the definitions in §1.4.4. For $f : X \to Y$ a morphism in AlgSpc we say that:

(a) f is *cohomologically pro-unipotent* if there is a presentation $X \xrightarrow{\sim} \lim_I X_i$ as a cofiltered limit where: each $X_i \to X_j$ is cohomologically unipotent affine and each $X_i \to Y$ is cohomologically unipotent;

(b) f is *essentially cohomologically pro-unipotent* if there is a factorization $X \to X' \to Y$ where $X \to X'$ is cohomologically pro-unipotent and $X' \to Y$ is finitely presented.

A morphism $f : \mathscr{X} \to \mathscr{Y}$ in PStk is said to be (resp. *essentially*) *cohomologically pro-unipotent* if for every affine scheme $S \to \mathscr{Y}$ the fiber product $S \underset{\mathscr{Y}}{\times} \mathscr{X}$ is a qcqs algebraic space and the induced morphism $S \underset{\mathscr{Y}}{\times} \mathscr{X} \to S$ is (resp. essentially) cohomologically pro-unipotent.

The following is the crucial result that will ultimately provide the base change for renormalized pushforwards that we need.

Proposition 1.21. *[19, Theorem 10.150] Consider the 3-functor formalism of (1.26). The class $E_{w.c.p.s.} \subset E_l$ of weakly cohomologically pro-smooth morphisms is right-adjointable with right adjoint f_*^{ren} that*

(i) is compatible with pullback with respect to the class $E_{e.c.u.}$ of essentially cohomologically unipotent morphisms, i.e. $(g_{\mathscr{Y}})^! \circ f_*^{\mathrm{ren}} \xrightarrow{\sim} (f')_*^{\mathrm{ren}} \circ (g_{\mathscr{X}})^!$ for $g_{\mathscr{Y}} \in E_{e.c.u.}$;

(ii) is compatible with the class $E_{ind-fp.}$ of representable ind-fp pushforward, i.e. $(g_{\mathscr{Y}})_* \circ (f')_*^{\mathrm{ren}} \xrightarrow{\sim} f_*^{\mathrm{ren}} \circ (g_{\mathscr{X}})_*$ for $g_{\mathscr{Y}} \in E_{ind-fp.}$;

(iii) satisfies the projection formula, i.e. $f_*^{\mathrm{ren}}(\mathscr{F}) \otimes \mathscr{G} \xrightarrow{\sim} f_*^{\mathrm{ren}}(\mathscr{F} \otimes f^!(\mathscr{G}))$ for every $\mathscr{F} \in D(\mathscr{X})$ and $\mathscr{G} \in D(\mathscr{Y})$.

The idea to proof the above Proposition is to check that one has left adjoints as in Proposition 1.9 ((iii)) for essentially cohomologically pro-unipotent morphisms, which allows one to extend the renormalized pushforward of Proposition 1.20 to weakly cohomologically pro-smooth morphisms between qcqs algebraic spaces. Then, since the category of ind-finite sheaves on ind-placid stacks satisfies universal descent also with respect to representable essentially cohomologically pro-unipotent morphisms one can lift the base change morphisms to this generality.

2. Newton strata as subschemes

2.1. **Newton strata: set-theoretic definition.** Recall the convetions of §1.1 (a). Let $\mathbf{G}$ be a connected reductive algebraic group over F. Let $\mathbf{S}$ be a maximal F-split torus of $\mathbf{G}$. Let $\mathbf{T}$ be the centralizer of $\mathbf{S}$ and $\mathbf{N_S}$ the normalizer of $\mathbf{S}$. Then we define the Iwahori-Weyl group $\breve{W} := \mathbf{N_S}(F)/\mathbf{T}(F)_1$, where $\mathbf{T}(F)_1$ is the kernel of the Kottwitz homomorphism $\mathbf{T}(F) \to X_*(\mathbf{N})_I$. The maximal split torus $\mathbf{S}$ determines an apartment $\mathscr{A}$ in the Bruhat–Tits building $\mathscr{B}(\mathbf{G}(F))$ associated to $\mathbf{G}(F)$. Let $\mathfrak{a}$ be the choice of an alcove (i.e. chamber) in $\mathscr{A}$ and let $\mathcal{I}_\mathfrak{a}$ denote the unique smooth group scheme over $\mathcal{O}_F$ with generic fiber $\mathbf{G}$ and such that $\mathcal{I}_\mathfrak{a}(\mathcal{O}_F)$ fixes $\mathfrak{a}$ in $\mathcal{B}(\mathbf{G}(F))$ (as constructed for instance, in [22, §8.4]). The choice of $\mathfrak{a}$ determines a splitting $\breve{W} = \breve{W}_a \rtimes \breve{\Omega}$, where W_a is the affine Weyl group and $\breve{\Omega}$ is the stabilizer of $\mathfrak{a}$ in $\mathscr{A}$ (see [22, §6.6.3]). Notice that $\breve{\Omega} \simeq \pi_1(\mathbf{G}(F))_I$ ([11, Lemma 14]), in particular $\breve{\Omega}$ does not depend on the choice of alcove $\mathfrak{a}$. We denote by $\breve{\mathbb{S}}$ the set of simple reflections of $\breve{W}_a$ and by $\breve{\ell}$ the length function on $\breve{W}$. We also have that $\breve{\Omega} = \ker \breve{\ell}$.

Notation 2.1. We will consider two situations:

(i) equal characteristic, i.e. $\operatorname{char} \mathbf{k} = \operatorname{char} F$. In this case, given a choice of uniformizer $\epsilon \in \mathcal{O}_F$ we have $\mathcal{O}_F \simeq \mathbf{k}[[\epsilon]]$ and $F \simeq \mathbf{k}((\epsilon))$. In particular, F is a $\mathbf{k}$-algebra. We let:

$$LG, \mathrm{Iw} : \mathrm{Aff} \longrightarrow \mathrm{Spc}, \qquad LG(A) := \mathbf{G}(A((\epsilon))), \qquad \mathrm{Iw}(A) := \mathcal{I}(A[[\epsilon]]). \tag{2.1}$$

(ii) mixed characteristic, i.e. $p = \operatorname{char} \mathbf{k} \neq \operatorname{char} F = 0$. In this case, we consider:

$$LG, \mathrm{Iw} : \mathrm{Aff}^{\mathrm{perf}} \longrightarrow \mathrm{Spc}, \qquad LG(A) := \mathbf{G}(W(A)[1/p]), \qquad \mathrm{Iw}(A) := \mathcal{I}(W_{\mathcal{O}_F}(A)), \tag{2.2}$$

where $W_{\mathcal{O}_F}(A) := W(A) \otimes_{W(\mathbf{k})} \mathcal{O}_F$, and, for a $\mathbf{k}$-algebra A, $W(A)$ denotes the ring of Witt vectors of A.

In both situations of Notation 2.1, we let $\breve{G} := LG(\mathbf{k}) = \mathbf{G}(\breve{F})$ and $\breve{I} := \mathrm{Iw}(\mathbf{k})$. It is well-known that we have the Cartan decomposition $\breve{G} = \sqcup_{w \in \breve{W}} \breve{I}\dot{w}\breve{I}$ (see [22, Theorem 5.2.1]). Let θ be a group automorphism on $\breve{G}$ with $\theta(\breve{I}) = \breve{I}$. Then the action of θ on $\breve{G}$ induces a length-preserving group automorphism on $\breve{W}$ (and hence induces a group automorphism on $\breve{\Omega}$ and a bijection on $\breve{\mathbb{S}}$). We denote these induced actions also by θ. We consider the θ-twisted conjugation action on $\breve{G}$ defined by $g \cdot_\theta g' = gg'\theta(g)^{-1}$. We define the θ-twisted conjugation action on $\breve{W}$ in the same way.

2.1.1. *Combinatorial input.* Following [17], we have two arithmetic invariants on the set of θ-twisted conjugacy classes of $\breve{W}$, given by the Kottwitz map and the Newton map. Note that the Kottwitz map and the Newton map do not distinguish all the θ-twisted conjugacy classes. However, they distinguish an important subfamily of θ-twisted conjugacy classes, called the straight θ-conjugacy classes.

An element x of $\breve{W}$ is called *θ-straight* if $\ell(x\theta(x)\cdots\theta^{n-1}(x)) = n\ell(x)$ for all positive integers n. A θ-twisted conjugacy class of $\breve{W}$ is called *straight* if it contains some θ-straight element. For any θ-straight element x, we denote by $\mathcal{O}_x$ the straight θ-conjugacy class containing x.

For $w, w' \in \breve{W}$ and $s \in \breve{\mathbb{S}}$, we write $w \xrightarrow{s}_\theta w'$ if $w' = sw\theta(s)$ and $\ell(w') \leqslant \ell(w)$. We write $w \to_\theta w'$ if there is a sequence $w = w_0, w_1, \cdots, w_n = w'$ of elements in $\breve{W}$ such that for any k, $w_{k-1} \xrightarrow{s}_\theta w_k$ for some $s \in \breve{\mathbb{S}}$. We write $w \approx_\theta w'$ if $w \to_\theta w'$ and $w' \to_\theta w$. It is easy to see that $w \approx_\theta w'$ if $w \to_\theta w'$ and $\ell(w) = \ell(w')$. For any θ-conjugacy class $\mathcal{O}$ in $\breve{W}$, we denote by $\mathcal{O}_{\min}$ the set of minimal length elements in $\mathcal{O}$. Now we recall some properties on the minimal length elements, obtained in [17, §2 and §3].

Theorem 2.2. *Let $\mathcal{O}$ be a θ-conjugacy class of $\breve{W}$ and $w \in \mathcal{O}$. Then there exists $w' \in \mathcal{O}_{\min}$ such that $w \to_\theta w'$.*

Theorem 2.3. *Let $\mathcal{O}$ be a straight θ-conjugacy class of $\breve{W}$ and $w, w' \in \mathcal{O}_{\min}$. Then there exists $\tau \in \breve{\Omega}$ such that $w \approx_\theta \tau w' \theta(\tau)^{-1}$.*

Let $\mathcal{O}$ be a straight θ-conjugacy class of $\breve{W}$. For $w \in \breve{W}$, we write $\mathcal{O} \preccurlyeq w$ if there exists a minimal length element $w' \in \mathcal{O}$ with $w' \leqslant w$. By [15, Proposition 2.4], if $\mathcal{O} \preccurlyeq w$ and $w_1 \to_\theta w$, then $\mathcal{O} \preccurlyeq w_1$. Note that for any given w, $\{w' \leqslant w\}$ is a finite set. Hence $\{\mathcal{O} \preccurlyeq w\}$ is again a finite set.

Let $\mathcal{O}_1, \mathcal{O}_2$ be straight θ-conjugacy classes of $\breve{W}$, we write $\mathcal{O}_1 \preccurlyeq \mathcal{O}_2$ if $\mathcal{O}_1 \preccurlyeq w$ for some minimal length element w of $\mathcal{O}_2$. By [15, §3.2], $\preccurlyeq$ is a partial order on the set of straight θ-conjugacy classes of $\breve{W}$.

2.1.2. *Set-theoretic Newton strata.* In [18], the first author and S. Nie introduced a "nice" decomposition of $\breve{G}$ into a disjoint union of subsets, called the Newton strata, such that

- each Newton stratum is stable under the θ-twisted conjugation action of $\breve{G}$;
- the Newton strata are indexed by a discrete set, which is a subset of the set of θ-conjugacy classes of $\breve{W}$.

The definition is as follows. Let $\mathcal{O}$ be a straight θ-conjugacy class of $\mathcal{O}$. We define $\breve{G}_{\mathcal{O}} = \breve{G} \cdot_\theta (\breve{I} \dot{w} \breve{I})$, where w is a minimal length element in $\mathcal{O}$. By [16, §3.2], $\breve{G}_{\mathcal{O}}$ is independent of the choice of the minimal length representatives of $\mathcal{O}$. We call $\breve{G}_{\mathcal{O}}$ the *Newton stratum* associated to $\mathcal{O}$. By [16, Theorem 3.2], we have the following Newton decomposition:

$$\breve{G} = \sqcup_{\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}} \breve{G}_{\mathcal{O}},$$

where $\breve{W} /\!\!/_\theta \breve{W}$ denotes the set of straight θ-conjugacy classes.

Following [14, §2.5], a subset $X \subseteq \breve{G}$ is *admissible* if for any $w \in \breve{W}$, there exists $n \in \mathbf{N}$ such that $X \cap \breve{I} \dot{w} \breve{I}$ is stable under the right action of $\breve{I}_n$.

2.2. **A technical property.** In this subsection, we establish the following technical result, which plays a crucial role in the constructions in §2.3.

For every w, we write $\overline{\breve{I} \dot{w} \breve{I}} = \cup_{w' \leq w} \breve{I} \dot{w}' \breve{I}$. For any $n \in \mathbb{N}$ and finite subset $\Gamma \subseteq \breve{\Omega}$ we let $\breve{G}^{\leqslant n}_{\Gamma} := \cup_{w \in \breve{W}_a \Gamma, \breve{\ell}(w) \leqslant n} \breve{I} \dot{w} \breve{I}$. We simply write $\breve{G}^{\leqslant n}$ for $\breve{G}^{\leqslant n}_{\breve{\Omega}}$. In particular, notice that $\breve{G}^{\leqslant n}_1 = \cup_{w \in \breve{W}_a, \breve{\ell}(w) \leqslant n} \breve{I} \dot{w} \breve{I}$ and $\breve{G}^{\leqslant n}_1 = \breve{G}^{\leqslant n} \cap \breve{G}_1$.

Proposition 2.4. *Let $w \in \breve{W}$. Then*

(1) $\breve{G} \cdot_\theta \overline{\breve{I} \dot{w} \breve{I}} = \sqcup_{\mathcal{O} \preccurlyeq w} \breve{G}_{\mathcal{O}}$.

(2) For any $w' \in \breve{W}$, there exists $n \in \mathbb{N}$ and a finite subset $\Gamma \subseteq \breve{\Omega}$ such that

$$\breve{G}_{\Gamma}^{\leqslant n} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}} \cap \breve{I}\dot{w}'\breve{I} = \breve{G} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}} \cap \breve{I}\dot{w}'\breve{I}.$$

Before giving the proof, we have the following immediate corollary.

Corollary 2.5. *Let w be a θ-straight element contained in a straight θ-conjugacy class $\mathcal{O}$. Then $\breve{G} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}} = \breve{G}_{\mathcal{O}} \sqcup (\cup_{w'<w} \breve{G} \cdot_\theta \overline{\breve{I}\dot{w}'\breve{I}})$.*

We also show that Proposition 2.4 implies the admissibility of the Newton strata of $\breve{G}$. Of course, this will also follow Theorem 2.7 (2) below (by noetherian approximation). But the proof given here is more elementary and more explicit.

Corollary 2.6. *For any straight θ-conjugacy class $\mathcal{O}$ of $\breve{W}$, $\breve{G}_{\mathcal{O}}$ is an admissible subset of $\breve{G}$.*

Proof. Let w be a θ-straight element in $\mathcal{O}$. By Proposition 2.4 (1), $\breve{G} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}} = \sqcup_{\mathcal{O}' \preccurlyeq w} \breve{G}_{\mathcal{O}'}$. By Proposition 2.4 (2), for any $w' \in \breve{W}$, there exists $n \in \mathbb{N}$, such that $\breve{I}\dot{w}'\breve{I} \cap (\sqcup_{\mathcal{O}' \preccurlyeq w} \breve{G}_{\mathcal{O}'}) = \breve{I}\dot{w}'\breve{I} \cap \breve{G}^{\leqslant n} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}}$. For any $g \in \breve{G}^{\leqslant n}$, $\theta(g)\breve{I}\theta(g)^{-1} \supset \breve{I}_n$. Thus $\breve{G}^{\leqslant n} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}}$ is stable under the right action of $\breve{I}_n$. Hence $\breve{I}\dot{w}'\breve{I} \cap (\sqcup_{\mathcal{O}' \preccurlyeq w} \breve{G}_{\mathcal{O}'})$ is stable under the right multiplication of $\breve{I}_n$. Similarly, $\breve{I}\dot{w}'\breve{I} \cap (\sqcup_{\mathcal{O}' \preccurlyeq w, \mathcal{O}' \neq \mathcal{O}} \breve{G}_{\mathcal{O}'})$ is stable under the right multiplication of $\breve{I}_{n'}$ for some $n' \in \mathbb{N}$. Thus $\breve{I}\dot{w}'\breve{I} \cap \breve{G}_{\mathcal{O}}$ is stable under the right multiplication of $\breve{I}_{\max\{n,n'\}}$. Hence $\breve{G}_{\mathcal{O}}$ is admissible. □

2.2.1. *Minimal length element case.* Suppose that w is of minimal length in its θ-conjugacy class. By [17, Proposition 2.7 & Theorem 2.9], there exists $J \subset \breve{\mathbb{S}}$ with W_J finite, a θ-straight element $x \in \breve{W}$ such that x is of minimal length in its $W_J \backslash \breve{W}/W_{\theta(J)}$-coset and $\mathrm{Ad}(x)\theta(J) = J$, and $u \in W_J$, such that $w \approx_\theta ux$. Let $\mathcal{O}$ be the θ-conjugacy class of x. Then $\mathcal{O}$ is straight. By definition, $\mathcal{O} \preccurlyeq ux$. By definition, $\mathcal{O} \preccurlyeq w$.

By definition, there exists a sequence $w = w_1 \xrightarrow{s_1}_\theta w_2 \xrightarrow{s_2}_\theta \cdots \xrightarrow{s_{n-1}}_\theta w_n = ux$, where $s_1, \dots, s_{n-1} \in \breve{\mathbb{S}}$, $w_1, \dots, w_n \in \breve{W}$ with $\ell(w) = \ell(w_1) = \cdots = \ell(w_{n-1}) = \ell(w_n) = \ell(ux)$. For any $i \leqslant n-1$, any element in $\breve{I}\dot{w}_i\breve{I}$ is θ-conjugated by an element in $\breve{I}\dot{s}_i\breve{I}$ to an element in $\breve{I}\dot{w}_{i+1}\breve{I}$.

Let $\breve{P}$ be the standard parahoric subgroup of $\breve{G}$ generated by $\breve{I}$ and $\dot{z}$ for $z \in W_J$. Let $U_{\breve{P}}$ be the pro-unipotent radical of $\breve{P}$ and $\overline{\breve{P}} = \breve{P}/U_{\breve{P}}$ be the reductive quotient of $\breve{P}$. Since $\mathrm{Ad}(x)\theta(J) = J$, the map $\bar{\theta}_x := \bar{p} \mapsto \dot{x}\theta(\bar{p})\dot{x}^{-1}$ gives an automorphism on $\overline{\breve{P}}$. Moreover, $\breve{I}/U_{\breve{P}}$ is a $\bar{\theta}_x$-stable Borel subgroup of $\overline{\breve{P}}$. By Steinberg's theorem, $\overline{\breve{P}} = \{\bar{p}\bar{p}'\bar{\theta}_x(\bar{p})^{-1}; \bar{p} \in \overline{\breve{P}}, \bar{p}' \in \breve{I}/U_{\breve{P}}\}$. Hence

$$\breve{I}\dot{u}\dot{x}\breve{I} \subset \breve{P}\dot{x} = \{pp'\dot{x}\theta(p)^{-1}; p \in \breve{P}, p' \in \breve{I}\}.$$

In other words, any element in $\breve{I}\dot{u}\dot{x}\breve{I}$ is θ-conjugated by an element in $\breve{P}$ to an element in $\breve{I}\dot{x}\breve{I}$.

Let w_J be the longest element in W_J. Then $\breve{P} = \sqcup_{z \in W_J} \breve{I}\dot{z}\breve{I} \subset \breve{G}^{\leqslant \ell(w_J)}$. We have

(a) $\breve{I}\dot{w}\breve{I} \subset \breve{G}_{\mathcal{O}}$.

(b) $\breve{I}\dot{w}\breve{I} \subset \breve{G}_1^{\leqslant (n+\ell(w_J))} \cdot_\theta (\breve{I}\dot{x}\breve{I})$.

Let x' be another θ-straight element in $\mathcal{O}$. Then by Theorem 2.3, $x \approx_\theta \tau x'\theta(\tau)^{-1}$ for some $\tau \in \breve{\Omega}$, that is $\tau x\theta(\tau)^{-1} \to_\theta x' \to_\theta \tau x\theta(\tau)^{-1}$. Hence there exists $n' \in \mathbb{N}$ such that $\breve{I}\dot{x}\breve{I} \subset \breve{G}_{\{\tau\}}^{\leqslant n'} \cdot_\theta (\breve{I}\dot{x}'\breve{I})$. Hence we have

(c) For any θ-straight element x' in $\mathcal{O}$, there exists $m \in \mathbb{N}$ and $\tau \in \breve{\Omega}$ such that $\breve{I}\dot{w}\breve{I} \subset \breve{G}^{\leqslant m}_{\{\tau\}} \cdot_\theta (\breve{I}\dot{x}'\breve{I})$.

2.2.2. *Inductive procedure.* We prove by induction on $\ell(w)$ that

(a) $\breve{I}\dot{w}\breve{I} \subset \sqcup_{\mathcal{O} \preccurlyeq w} \breve{G}_{\mathcal{O}}$.

(b) For any θ-straight element x, there exists $m \in \mathbb{N}$ and a finite subset $\Gamma \subseteq \breve{\Omega}$ such that

$$\breve{I}\dot{w}\breve{I} \cap \breve{G}_{\mathcal{O}_x} \subset \breve{G}^{\leqslant m}_{\Gamma} \cdot_\theta (\breve{I}\dot{x}\breve{I}).$$

The case where w is a minimal length element in its θ-conjugacy class is established in §2.2.1. Now suppose that w is not of minimal length in its θ-conjugacy class. By Theorem 2.2, there exists a sequence $w = w_1 \xrightarrow{s_1}_\theta w_2 \xrightarrow{s_2}_\theta \cdots \xrightarrow{s_{n-1}}_\theta w_n$, where $s_1, \dots, s_{n-1} \in \breve{\mathbb{S}}$, $w_1, \dots, w_n \in \breve{W}$ with $\ell(w) = \ell(w_1) = \cdots = \ell(w_{n-1}) > \ell(w_n)$. For any $i < n-1$, any element in $\breve{I}\dot{w}_i\breve{I}$ is θ-conjugated by an element in $\breve{I}\dot{s}_i\breve{I}$ to an element in $\breve{I}\dot{w}_{i+1}\breve{I}$.

Similarly, any element in $\breve{I}\dot{w}_{n-1}\breve{I}$ is conjugate by an element in $\breve{I}\dot{s}_{n-1}\breve{I}$ to an element in $\breve{I}\dot{w}_n\breve{I} \sqcup \breve{I}\dot{w}'_n\breve{I}$, where $w'_n = s_{n-1}w_{n-1}$. Therefore $\breve{I}\dot{w}\breve{I} \subset \breve{G}^{\leqslant n} \cdot_\theta (\breve{I}\dot{w}_n\breve{I} \sqcup \breve{I}\dot{w}'_n\breve{I})$.

Note that $\ell(w_n) = \ell(w) - 2$ and $\ell(w'_n) = \ell(w) - 1$. Note that $\ell(w_n), \ell(w'_n) < \ell(w_{n-1}) = \ell(w)$. By inductive hypothesis, $\breve{G} \cdot_\theta (\breve{I}\dot{w}_n\breve{I}) \cup \breve{G} \cdot_\theta (\breve{I}\dot{w}'_n\breve{I}) \subset \sqcup_{\mathcal{O} \preccurlyeq w_n \text{ or } \mathcal{O} \preccurlyeq w'_n} \breve{G}_{\mathcal{O}}$. We have $w_n < w_{n-1}$ and $w'_n < w_{n-1}$. Hence $\mathcal{O} \preccurlyeq w_n$ or $\mathcal{O} \preccurlyeq w'_n$ implies that $\mathcal{O} \preccurlyeq w_{n-1}$. By [15, Proposition 2.4], $\mathcal{O} \preccurlyeq w$. So $\breve{I}\dot{w}\breve{I} \subset \sqcup_{\mathcal{O} \preccurlyeq w} \breve{G}_{\mathcal{O}}$.

Let x be a θ-straight element with $\mathcal{O}_x \preccurlyeq w$. By inductive hypothesis, there exists $n' \in \mathbb{N}$ and finite subsets $\Gamma_1, \Gamma_2 \subseteq \breve{\Omega}$ such that $\breve{I}\dot{w}_n\breve{I} \cap \breve{G}_{\mathcal{O}_x} \subset \breve{G}^{\leqslant n'}_{\Gamma_1} \cdot_\theta (\breve{I}\dot{x}\breve{I})$ and $\breve{I}\dot{w}'_n\breve{I} \cap \breve{G}_{\mathcal{O}_x} \subset \breve{G}^{\leqslant n'}_{\Gamma_2} \cdot_\theta (\breve{I}\dot{x}\breve{I})$. Since $\breve{I}\dot{w}\breve{I} \subset \breve{G}^{\leqslant n}_1 \cdot_\theta (\breve{I}\dot{w}_n\breve{I} \sqcup \breve{I}\dot{w}'_n\breve{I})$, we have $\breve{I}\dot{w}\breve{I} \cap \breve{G}_{\mathcal{O}_x} \subset \breve{G}^{\leqslant (n+n')}_{\Gamma_1 \cup \Gamma_2} \cdot_\theta (\breve{I}\dot{x}\breve{I})$.

2.2.3. *Proof of Proposition 2.4.* By §2.2.2 (a), for any $y \leqslant w$, we have that $\breve{G} \cdot_\theta (\breve{I}\dot{y}\breve{I}) \subset \sqcup_{\mathcal{O} \preccurlyeq y} \breve{G}_{\mathcal{O}}$. By definition, $\mathcal{O} \preccurlyeq y$ implies that $\mathcal{O} \preccurlyeq w$. Hence $\overline{\breve{I}\dot{w}\breve{I}} = \sqcup_{y \leqslant w} \breve{I}\dot{y}\breve{I} \subset \sqcup_{\mathcal{O} \preccurlyeq w} \breve{G}_{\mathcal{O}}$. On the other hand, if $\mathcal{O} \preccurlyeq w$, then there exists a minimal length element y of $\mathcal{O}$ with $y \leqslant w$. By definition, $\breve{G}_{\mathcal{O}} = \breve{G} \cdot_\theta (\breve{I}\dot{y}\breve{I})$. Then $\sqcup_{\mathcal{O} \preccurlyeq w} \breve{G}_{\mathcal{O}} \subset \cup_{y \leqslant w} \breve{G} \cdot_\theta (\breve{I}\dot{y}\breve{I}) = \breve{G} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}}$. This finishes the proof of part (1).

Let $\mathcal{O} \preccurlyeq w$ and x be a θ-straight element in $\mathcal{O}$. By definition, there exists a minimal length element x' of $\mathcal{O}_x$ with $x' \leqslant w$. Since x is θ-straight by §2.2.1, there exists $k \in \mathbb{N}$ and $\tau \in \breve{\Omega}$ such that $\breve{I}\dot{x}\breve{I} \subset \breve{G}^{\leqslant k}_{\{\tau\}} \cdot_\theta (\breve{I}\dot{x}'\breve{I}) \subset \breve{G}^{\leqslant k}_{\{\tau\}} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}}$. By §2.2.2, there exists $m \in \mathbb{N}$ and a finite subset $\Gamma \subseteq \breve{\Omega}$ such that $\breve{I}\dot{w}'\breve{I} \cap \breve{G}_{\mathcal{O}} \subset \breve{G}^{\leqslant m}_{\Gamma} \cdot_\theta (\breve{I}\dot{x}\breve{I})$. Then $\breve{I}\dot{w}'\breve{I} \cap \breve{G}_{\mathcal{O}} \subset \breve{G}^{\leqslant (k+m)}_{\Gamma'} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}}$, where $\Gamma' = \{z\tau \,;\, z \in \Gamma\}$. As $\{\mathcal{O} \preccurlyeq w\}$ is a finite set, there exists $n \in \mathbb{N}$ and a finite subset $\Gamma' \subseteq \breve{\Omega}$ such that

$$\breve{G} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}} \cap \breve{I}\dot{w}'\breve{I} = \sqcup_{\mathcal{O} \preccurlyeq w} \breve{I}\dot{w}'\breve{I} \cap \breve{G}_{\mathcal{O}} \subset \breve{G}^{\leqslant n}_{\Gamma'} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}}.$$

This finishes the proof of part (2).

2.3. **Newton strata: algebro-geometric definition.** The main goal of the remaining of this section is to prove the following theorem.

Theorem 2.7. *Let $\mathcal{O}$ be a straight θ-conjugacy class of $\breve{W}$. Then there is a (perfectly) finitely presented locally closed embedding $\imath_{\mathcal{O}} : LG_{\mathcal{O}} \hookrightarrow LG$ with $LG_{\mathcal{O}}$ reduced such that*

(1) $\imath_{\mathcal{O}}$ factors as:

$$LG_{\mathcal{O}} \xhookrightarrow{\jmath_{\mathcal{O}}} LG_{\overline{\mathcal{O}}} \quad \text{and} \quad LG_{\overline{\mathcal{O}}} \xhookrightarrow{\imath_{\overline{\mathcal{O}}}} LG \tag{2.3}$$

where $\imath_{\overline{\mathcal{O}}}$ is a (perfecly) finitely presented closed embedding, and $\jmath_{\mathcal{O}}$ is a quasi-compact open embedding with dense image;

(2) $LG_{\mathcal{O}}(\mathbf{k}) = \breve{G}_{\mathcal{O}}$ and $LG_{\overline{\mathcal{O}}}(\mathbf{k}) = \cup_{\mathcal{O}' \preccurlyeq \mathcal{O}} \breve{G}_{\mathcal{O}'}$;

(3) We have $LG_{\mathrm{red}} = \mathrm{colim}_{\mathcal{O}}\, LG_{\overline{\mathcal{O}}}$.

We note that as we are outside the traditional finite type algebraic geometry, $\mathbf{k}$-points are usually not enough to determine the underlying schemes. So the theorem is more subtle than naive thought. The crucial point is finite presentation of $\jmath_{\mathcal{O}}$ and $\imath_{\overline{\mathcal{O}}}$. Once this is available, then $LG_{\mathcal{O}}$ is a placid ind-scheme, and the second condition of the theorem uniquely determines $LG_{\mathcal{O}} \subset LG_{\overline{\mathcal{O}}}$ as locally closed subspaces of LG.

2.3.1. *The closed scheme $LG_{\leqslant [w]_\theta}$.* Let $\mathrm{Fl} := LG/\mathrm{Iw}$ be the affine flag variety of $\mathbf{G}$. For each $w \in \breve{W}$, let

$$\mathrm{Fl}_w \xhookrightarrow{j_w} \mathrm{Fl}_{\leqslant w} \xhookrightarrow{i_{\leqslant w}} \mathrm{Fl}$$

be the corresponding (locally) closed embedding of Schubert cell (variety). We write $i_w = i_{\leqslant w} \circ j_w$.

Remark 2.8. Notice that since Fl is an ind-scheme of finite type ([32, Theorem 1.4]), there are at least two ways we can define the Schubert cell and variety:

(1) define Fl_w as the reduced subscheme with underlying set $\breve{I}\dot{w}$, then let $\mathrm{Fl}_{\leqslant w}$ be the reduced subscheme of Fl corresponding to the Zariski closure of Fl_w;

(2) define $\mathrm{Fl}_{\leqslant w}$ to be the scheme theoretic image of $\mathrm{Iw} \overset{(-)\cdot \dot{w}}{\to} LG \to \mathrm{Fl}$, then let Fl_w be the open complement of $\sqcup_{w' < w} \mathrm{Fl}_{w'} \hookrightarrow \mathrm{Fl}_w$.

When Fl is reduced, e.g. if $\mathbf{G}$ is semi-simple, splits over a tamely ramified extension of F and $p \nmid \pi_1(\mathbb{G})$ ([32, Theorem 0.2]), then both definitions agree. In the context of Newton strata, it is the second definition that generalizes, but in a subtle way.

Let $LG_{(\leqslant)w}$ be the inverse image of $\mathrm{Fl}_{(\leqslant)w}$ under the projection $LG \to \mathrm{Fl}$. Then $LG_{(\leqslant)w}$ is an Iw-torsor over $\mathrm{Fl}_{(\leqslant)w}$, and therefore is a qcqs scheme. In addition, $LG_{(\leqslant)w} \to LG$ is a fp (locally) closed embedding, still denoted by $i_{(\leqslant w)}$, and the inclusion $LG_w \subset LG_{\leqslant w}$ is affine, still denoted by j_w. Note that $LG_w(\mathbf{k}) = \breve{I}\dot{w}\breve{I}$ and $LG_{\leq w}(\mathbf{k}) = \overline{\breve{I}\dot{w}\breve{I}}$.

Write $LG = \mathrm{colim}_i\, S_i$ with S_i affine and $S_i \to S_j$ closed embedding, then $LG_{\leqslant w} \to LG$ factors through some $LG_{\leqslant w} \to S_i$. So $LG_{\leqslant w}$ is in fact affine, and so is LG_w. We also have $LG_{\mathrm{red}} = \mathrm{colim}_w\, LG_{\leqslant w}$, as $\mathrm{Fl}_{\mathrm{red}} = \mathrm{colim}_w\, \mathrm{Fl}_{\leqslant w}$.

Now let $\theta : LG \to LG$ be a lift of the automorphism $\theta : \breve{G} \to \breve{G}$ to the loop group ind-scheme which preserves Iw. The first goal is to associate to every $w \in \breve{W}$ another ind-scheme $LG_{\leqslant [w]_\theta}$ such that the inclusion map

$$i_{\leqslant [w]_\theta} : LG_{\leqslant [w]_\theta} \longrightarrow LG_{\mathrm{red}}$$

is still a fp closed embedding. Informally, $LG_{\leqslant [w]_\theta}$ is the LG-orbit in LG of $LG_{\leqslant w}$ under the θ-conjugation action. Here is the precise algebro-geometric construction.

Consider the following commutative diagram

(2.4)

$$
\begin{tikzcd}
LG_{\leqslant v} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leqslant w} \arrow[r, "i_{\leqslant v}"] \arrow[d, "i_{\leqslant w}"'] & LG \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leqslant w} \arrow[rrr] \arrow[d, "i_{\leqslant w}"'] & & & \frac{LG_{\leqslant w}}{\mathrm{Ad}_\theta \mathrm{Iw}} \arrow[d, "i_{\leqslant w}"] \\
LG_{\leqslant v} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG \arrow[r, "i_{\leqslant v}"] \arrow[d, "\beta_{\leqslant v}"', "\cong"] & LG \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG \arrow[r, "\alpha", "\cong"'] \arrow[d, "\beta"', "\cong"] & LG \times^{\mathrm{Iw}} LG \arrow[r] \arrow[d, "m"'] & \frac{LG \times^{\mathrm{Iw}} LG}{\mathrm{Ad}_\theta \mathrm{Iw}} \arrow[r, "\overrightarrow{h}"] \arrow[d, "\overleftarrow{h}"] & \frac{LG}{\mathrm{Ad}_\theta \mathrm{Iw}} \arrow[d, "\mathrm{Nt}"] \\
\mathrm{Fl}_{\leqslant v} \times LG \arrow[r, "i_{\leqslant v}"] & \mathrm{Fl} \times LG \arrow[r, "\mathrm{pr}"] & LG \arrow[r] & \frac{LG}{\mathrm{Ad}_\theta \mathrm{Iw}} \arrow[r, "\mathrm{Nt}"] & \frac{LG}{\mathrm{Ad}_\theta LG},
\end{tikzcd}
$$

where

- $\overleftarrow{h}$ (resp. $\overrightarrow{h}$) is induced by $LG \times LG \to LG$, $(g_1, g_2) = g_1 g_2$ (resp. $(g_1, g_2) = g_2\theta(g_1)$);
- α is induced by $LG \times LG \to LG \times LG$, $(g_1, g_2) = (g_1, g_2\theta(g_1)^{-1})$;
- β is induced by $LG \times LG \to LG \times LG$, $(g_1, g_2) = (g_1, g_1 g_2\theta(g_1)^{-1})$.

Note that except for the square giving $m \circ \alpha = \mathrm{pr} \circ \beta$, all other commutative squares are Cartesian. In addition, by étale descent of affine morphisms, $LG_{\leqslant v} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{(\leqslant)w}$ is affine over $\mathrm{Fl}_{\leqslant v}$, and therefore is a qcqs scheme.

It follows that the composed morphism (along the left column and the bottom arrow)

$$\mathrm{Ad}_\theta^{v,w} : LG_{\leqslant v} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leqslant w} \longrightarrow LG, \quad (g_1, g_2) \longmapsto g_1 g_2 \theta(g_1)^{-1}$$

is fp proper.

For each $m \in \mathbf{N}$ and finite subset $\Gamma \subseteq \breve{\Omega}$ we let $LG_\Gamma^{\leqslant m} := \cup_{z \in \breve{W}\Gamma,\, \ell(z) \leqslant m} LG_{\leqslant z}$. The morphism:

$$\mathrm{Ad}_\theta^{m,\Gamma,w} : LG_\Gamma^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leqslant w} \longrightarrow LG$$

is also fp proper. This means that for every closed embedding $i_{\leqslant u} : LG_{\leqslant u} \to LG$, the base changed morphism

$$\mathrm{Ad}_\theta^{u,m,\Gamma,w} : LG_{\leqslant u} \times_{LG} (LG_\Gamma^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leqslant w}) \longrightarrow LG_{\leqslant u}$$

is a fp proper morphism of qcqs schemes. By noetherian approximation ([35, Lemma 01ZM]), this morphism arises as the base change of a proper morphism

$$C_{u,m,\Gamma,w}^{(n)} \longrightarrow \mathrm{Fl}_{\leqslant u}^{(n)} := LG_{\leqslant u}/\mathrm{Iw}^{(n)} \tag{2.5}$$

between finite type schemes over $\mathbf{k}$, where $n \gg 0$ and $\mathrm{Iw}^{(n)}$ is the nth principal congruence subgroup of Iw. (So $\mathrm{Fl}_{\leqslant u}^{(n)} \to \mathrm{Fl}_{\leqslant u}$ is a $\mathrm{Iw}/\mathrm{Iw}^{(n)}$-torsor.)

Such morphism necessarily factors as

$$C_{u,m,\Gamma,w}^{(n)} \longrightarrow Z_{u,m,\Gamma,w}^{(n)} \subset \mathrm{Fl}_{\leqslant u}^{(n)}$$

where $Z_{u,m,\Gamma,w}^{(n)}$ is the scheme theoretic image (see [35, Definition 01R7]) of (2.5). Then $C_{u,m,\Gamma,w}^{(n)} \to Z_{u,m,\Gamma,w}^{(n)}$ is proper surjective and $Z_{u,m,\Gamma,w}^{(n)} \subset \mathrm{Fl}_{\leqslant u}^{(n)}$ is a closed embedding. As the scheme theoretic image of quasi-compact morphisms commute with flat base change ([35, Lemma 081I]), we see that the schematic image of $\mathrm{Ad}_\theta^{u,m,\Gamma,w}$, denoted by $Z_{u,m,\Gamma,w}$, is the base change of $Z_{u,m,\Gamma,w}^{(n)}$ along the projection $LG_{\leqslant u} \to \mathrm{Fl}_{\leqslant u}^{(n)}$. In particular, $\mathrm{Ad}_\theta^{u,m,\Gamma,w}$ factors as

$$LG_{\leqslant u} \times_{LG} (LG_\Gamma^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leqslant w}) \longrightarrow Z_{u,m,\Gamma,w} \subset LG_{\leqslant u}$$

with the first morphism surjective fp proper and the second morphism fp closed embedding.

Now if $m \leqslant m'$ and $\Gamma \subseteq \Gamma'$ we have the following commutative diagram

$$\begin{tikzcd} LG_{\leqslant u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leqslant w}) \arrow[r] \arrow[d] & Z_{u,m,\Gamma,w} \arrow[r] \arrow[d] & LG_{\leqslant u} \arrow[d, equal] \\ LG_{\leqslant u} \times_{LG} (LG_{\Gamma'}^{\leqslant m'} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leqslant w}) \arrow[r] & Z_{u,m',\Gamma',w} \arrow[r] & LG_{\leqslant u} \end{tikzcd}$$

As the left vertical map is a closed embedding, so is the middle vertical map. In addition, it is fp. We need the following lemma.

Lemma 2.9. *Fix u, w. Then there is some $m \geqslant 0$ and finite $\Gamma \subseteq \breve{\Omega}$ such that $Z_{u,m,\Gamma,w} \to Z_{u,m',\Gamma',w}$ is a nilpotent thickening for every $m' \geqslant m$ and $\Gamma' \supseteq \Gamma$.*

Proof. We first note that as $\mathbf{k}$-points lift along any fp morphisms, we have

$$Z_{u,m,\Gamma,w}(\mathbf{k}) = \overline{\breve{I}u\breve{I}} \cap (\breve{G}_{\Gamma}^{\leqslant m} \cdot_\theta \overline{\breve{I}w\breve{I}}). \tag{2.6}$$

Then by Proposition 2.4 (2), for fixed u, w, there exists some $m \in \mathbb{N}$ and finite $\Gamma \subseteq \breve{\Omega}$ such that

$$\overline{\breve{I}u\breve{I}} \cap (\breve{G}_{\Gamma}^{\leqslant m} \cdot_\theta \overline{\breve{I}w\breve{I}}) = Z_{u,m,\Gamma,w}(\mathbf{k}) \xrightarrow{\sim} Z_{u,m',\Gamma',w}(\mathbf{k}) = \overline{\breve{I}u\breve{I}} \cap (\breve{G}_{\Gamma'}^{\leqslant m'} \cdot_\theta \overline{\breve{I}w\breve{I}}) \tag{2.7}$$

for every $m' \geqslant m$ and $\Gamma' \supseteq \Gamma$. Now as both morphisms $Z_{u,m,\Gamma,w} \to Z_{u,m',\Gamma',w} \to LG_{\leqslant u}$ are fp closed embeddings, they arise as the pullback of morphisms $Z_{u,m,\Gamma,w}^{(n)} \to Z_{u,m',\Gamma',w}^{(n)} \to LG_{\leqslant u}/\mathrm{Iw}^{(n)}$ between finite type schemes over $\mathbf{k}$. Notice that (2.7) implies that $Z_{u,m,\Gamma,w}^{(n)}(\mathbf{k}) \xrightarrow{\sim} Z_{u,m',\Gamma',w}^{(n)}(\mathbf{k})$, so we see that $Z_{u,m,\Gamma,w}^{(n)} \to Z_{u,m',\Gamma',w}^{(n)}$ is a nilpotent thickening. Therefore, $Z_{u,m,\Gamma,w} \to Z_{u,m',\Gamma',w}$ is also a nilpotent thickening. □

The above lemma implies that

$$LG_{\leqslant u, \leqslant [w]_\theta} := \operatorname*{colim}_{m \geq 0,\, \Gamma \subseteq \breve{\Omega}} (Z_{u,m,\Gamma,w})_{\mathrm{red}} \subset LG_{\leqslant u}$$

is a fp closed embedding. In particular $LG_{\leqslant u,\leqslant [w]_\theta}$ is a qcqs scheme. E.g. $LG_{\leqslant w,\leqslant [w]_\theta} = LG_{\leqslant w}$. Finally, let

$$LG_{\leqslant [w]_\theta} := \operatorname*{colim}_{u \in \breve{W}} LG_{u,\leqslant [w]_\theta} \subset LG. \tag{2.8}$$

This is an ind-scheme, with the inclusion morphism $LG_{\leqslant [w]_\theta} \to LG$ being fp closed embedding, as desired. By (2.6), we have

$$LG_{\leqslant [w]_\theta}(\mathbf{k}) = \breve{G} \cdot_\theta \overline{\breve{I}\dot{w}\breve{I}}.$$

The construction also shows that $LG_{\leqslant [w]_\theta}$ is ind-placid. Indeed, $LG_{\leqslant u,\leqslant [w]_\theta} = \bigcup_{m \geq 0} (Z_{u,m,\Gamma,w})_{\mathrm{red}}$ for finitely many m and $\Gamma \in \breve{\Omega}$, and is an $\mathrm{Iw}^{(n)}$-torsor over $\bigcup_{m \geq 0} (Z_{u,m,\Gamma,w})_{\mathrm{red}}^{(n)}$ for some n, and therefore is placid. Furthermore, it is stable under the θ-conjugation action of LG on itself. We also have $LG_{\leqslant [w]_\theta} \subset LG_{\leqslant [w']}$ if $w \leqslant w'$, and

$$LG_{\mathrm{red}} = \bigcup_{w \in \breve{W}} LG_{\leqslant [w]_\theta}. \tag{2.9}$$

2.3.2. *The locally closed scheme $LG_{[w]_\theta}$.* Now we fix a straight θ-conjugacy class and let w be a θ-straight element in that conjugacy class.

We let

$$LG_{[w]_\theta} = LG_{\leqslant [w]_\theta} - \bigcup_{w'<w} LG_{\leqslant [w']_\theta}.$$

Then $LG_{[w]_\theta} \subset LG_{\leqslant [w]_\theta}$ is quasi-compact open embedding, and therefore $LG_{[w]_\theta}$ is a placid ind-scheme as well. By Corollary 2.5, we have

$$LG_{[w]_\theta}(\mathbf{k}) = \breve{G}_{\mathcal{O}}.$$

Next, we prove that

$$LG_{[w]_\theta} = LG_{[w']_\theta}$$

if w and w' are of two straight elements in the same θ-conjugacy class of $\breve{W}$. For this, it is enough to show that $LG_{[w]_\theta} \cap LG_{\leqslant u} = LG_{[w']_\theta} \cap LG_{\leqslant u}$ for every u. As both $LG_{[w]_\theta} \cap LG_{\leqslant u} \to LG_{\leqslant u}$ and $LG_{[w']_\theta} \cap LG_{\leqslant u} \to LG_{\leqslant u}$ are fp locally closed embedding, they arise as base change of some locally closed reduced subschemes of $\mathrm{Fl}^{(n)}_{\leqslant u}$. It then is enough to show that $LG_{[w]_\theta}(\mathbf{k}) \cap LG_{\leqslant u}(\mathbf{k}) = LG_{[w']_\theta}(\mathbf{k}) \cap LG_{\leqslant u}(\mathbf{k})$. But both are just $LG_{\leqslant u}(\mathbf{k}) \cap \breve{G}_{\mathcal{O}}$.

Now for every θ-straight conjugacy class $\mathcal{O} \subset \breve{W}$, we can define

$$LG_{\mathcal{O}} = LG_{[w]_\theta},$$

for some θ-straight element w in this conjugacy class.

We need one more lemma.

Lemma 2.10. *For every u, the composed morphism*

$$LG_{\leq u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_w) \subset LG_{\leq u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leq w}) \longrightarrow LG_{\leqslant u} \cap LG_{\leq [w]_\theta}$$

factors as

$$LG_{\leq u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_w) \longrightarrow LG_{\leq u} \cap LG_{\mathcal{O}} \subset LG_{\leqslant u} \cap LG_{\leq [w]_\theta}.$$

Proof. We consider the map

$$LG_{\leq u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_w) \times_{LG_{\leqslant u} \cap LG_{\leq [w]_\theta}} LG_{\leq u} \cap LG_{\mathcal{O}} \subset LG_{\leq u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_w),$$

which is a quasi-compact open embedding (as it is the base change of $LG_{\leq u} \cap LG_{\mathcal{O}} \subset LG_{\leq u} \cap LG_{\leq [w]_\theta}$). By Corollary 2.5, the above map induces a bijection on $\mathbf{k}$-points. Therefore it is an isomorphism. The lemma then follows. ∎

2.3.3. *Proof of Theorem 2.7.* Given all the discussions above, the remaining key point is to show that for $w \in \mathcal{O}$ being θ-straight, we have

$$\overline{LG_{\mathcal{O}}} = LG_{\leqslant [w]_\theta}.$$

This means that every point $x \in LG_{\leqslant [w]_\theta}$ admits a generalization to a point $\eta \in LG_{[w]_\theta}$. We may lift x to a point x' to $LG_{\leq u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leq w})$. As $LG_w \subset LG_{\leqslant w}$ is open dense, the embedding $LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_w \subset LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leq w}$ is also open dense. Therefore, after possibly enlarging u, x' admits a generalization to a point $\eta' \in LG_{\leq u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_w)$. I.e. there is a valuation ring V and a map $\operatorname{Spec} V \to LG_{\leq u} \times_{LG} (LG_{\Gamma}^{\leqslant m} \times^{\mathrm{Iw},\mathrm{Ad}_\theta} LG_{\leq w})$ such that the generic point of $\operatorname{Spec} V$ maps to η' and the closed point of $\operatorname{Spec} V$ maps to x'. The image η of η' in $LG_{\leqslant u} \cap LG_{\leqslant [w]_\theta}$ actually belongs to $LG_{\leqslant u} \cap LG_{\mathcal{O}}$ by Lemma 2.10. In other words, we have a map $\operatorname{Spec} V \to LG_{\leq [w]_\theta}$, which sends the closed point to the given point x and sends the generic point to $\eta \in LG_{\leq u} \cap LG_{\mathcal{O}}$. It follows that $\overline{LG_{\mathcal{O}}} = LG_{\leq [w]_\theta}$.

In particular, we see that $LG_{\leq [w]_\theta}$ is independent of the choice of minimal lenght elements in a straight θ-conjugacy class and we can define

$$LG_{\overline{\mathcal{O}}} := LG_{\leq [w]_\theta} = \overline{LG_{\mathcal{O}}}.$$

Then it follows from Proposition 2.4 (1) and $LG_{\overline{\mathcal{O}}}(\mathbf{k}) = \cup_{\mathcal{O}' \preccurlyeq \mathcal{O}} LG_{\mathcal{O}'}(\mathbf{k})$. The first two parts of the theorem thus have been proved.

Recall that $LG_{\mathrm{red}} := \mathrm{colim}_{i \in I} (X_i)_{\mathrm{red}}$, for any presentation $LG = \mathrm{colim}_I X_i$ as an ind-scheme. Thus, the last part of the theorem follows from (2.9).

3. Newton decomposition of $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$

3.1. **Categorical closed stratifications from geometric stratifications.** In this section we prove the following general result that shows that a stratification of a sifted-placid stack $\mathscr{X}$ by fp-closed stratum produces a closed stratification of the category of sheaves $D(\mathscr{X})$.

Proposition 3.1. *Let P be a down-finite partially ordered set and assume we are given a collection $\{\mathscr{X}_p\}_{p \in \mathsf{P}}$ of sifted-placid stacks, such that*

(1) the closure $\overline{\mathscr{X}_p} \simeq \mathscr{X}_{\overline{p}} := \sqcup_{q \leqslant p} \mathscr{X}_q$, $\jmath_p : \mathscr{X}_p \hookrightarrow \mathscr{X}_{\overline{p}}$ is qc qs and $\imath_{\overline{p}} : \mathscr{X}_{\overline{p}} \hookrightarrow \mathscr{X}$ is finitely presented closed;

(2) one has $\mathrm{colim}_{p \in \mathsf{P}} \overline{\mathscr{X}_p} = \mathscr{X}_{\mathrm{red}}$.

Then the assignment:

$$\mathsf{P} \longrightarrow \mathrm{Cl}_{D(\mathscr{X})}, \qquad p \longmapsto (\imath_{\overline{p}})_* : D(\mathscr{X}_{\overline{p}}) \hookrightarrow D(\mathscr{X})$$

gives a P-closed stratification of $D(\mathscr{X})$.

Before giving the proof of Proposition 3.1 we need the following formal consequence of Lemma 1.18.

Lemma 3.2. *Let*

$$\begin{tikzcd} Y \arrow[r, hook, "k_2"] \arrow[d, hook, "k_1"'] & Z_2 \arrow[d, hook, "i_2"] \\ Z_1 \arrow[r, hook, "i_1"'] & X \end{tikzcd} , \tag{3.1}$$

be a pullback square of fp-closed embeddings of sifted-placid stacks, where $Y = Z_1 \cap Z_2$ and $X = Z_1 \cup Z_2$. Then

$$\begin{tikzcd} D(Y) \arrow[r, hook, "k_{2,*}"] \arrow[d, hook, "k_{1,*}"'] & D(Z_2) \arrow[d, hook, "i_{2,*}"] \\ D(Z_1) \arrow[r, hook, "i_{1,*}"'] & D(X) \end{tikzcd}$$

is a push-out of categories in $\mathrm{Pr}^{\mathrm{R}}_{/D(X)}$.

Proof. Let $j_1 : U_1 := X\backslash Z_1 \hookrightarrow X$ denote the open complement of Z_1. Notice that $Z_2\backslash Y = U_1$ and let $l_2 : U_1 \hookrightarrow Z_2$ denote the open complement to k_2. By the open-closed gluing of constructible sheaves we have an equivalence:

$$D(X) \xrightarrow{\sim} \overset{\text{l.lax}}{\lim}\left(D(U_1) \overset{i_1^*j_{1,*}}{\longrightarrow} D(Z_1)\right)$$
$$\mathscr{F} \longmapsto (j_1^!\mathscr{F}, i_1^!\mathscr{F}, g_{\mathscr{F}} : i_1^*j_{1,*}j_1^!\mathscr{F} \longrightarrow i_1^!\mathscr{F}[1]),$$

where the morphism $g_{\mathscr{F}}$ is induced by the cofiber-fiber sequence $i_{1,*}i_1^!\mathscr{F} \to \mathscr{F} \to j_{1,*}j_1^!\mathscr{F}$. Here the category $\lim^{\text{l.lax}}\left(D(U_1) \overset{i_1^*j_{1,*}}{\to} D(Z_1)\right)$ has a concrete description as triples

$$(\mathscr{F}_{U_1} \in D(U_1), \mathscr{F}_{Z_1} \in D(Z_1), \alpha : i_1^*j_{1,*}\mathscr{F}_{U_1} \longrightarrow \mathscr{F}_{Z_1}[1]).$$

We also have an equivalence:

$$D(Z_2) \xrightarrow{\sim} \overset{\text{l.lax}}{\lim}\left(D(U_1) \overset{k_2^*l_{2,*}}{\longrightarrow} D(Y)\right)$$
$$\mathscr{G} \longmapsto (l_2^!\mathscr{G}, k_2^!\mathscr{G}, g_{\mathscr{G}} : k_2^*l_{2,*}l_2^!\mathscr{G} \longrightarrow k_2^!\mathscr{G}[1]),$$

where $g_{\mathscr{G}}$ is induced by the cofiber-fiber sequence $k_{2,*}k_2^!\mathscr{G} \to \mathscr{G} \to l_{2,*}l_2^!\mathscr{G}$ and the category $\lim^{\text{l.lax}}\left(D(U_1) \overset{k_2^*l_{2,*}}{\to} D(Y)\right)$ can be concretely described as the category of triples:

$$(\mathscr{G}_{U_1} \in D(U_1), \mathscr{G}_Y \in D(Y), \alpha : k_2^*l_{2,*}\mathscr{G}_{U_1} \longrightarrow \mathscr{G}_Y[1]).$$

Now we claim that

$$\Psi : D(X) \longrightarrow D(Z_1) \bigsqcup_{D(Y)} D(Z_2) \tag{3.2}$$
$$\mathscr{F} \longmapsto (i_1^!\mathscr{F}, i_2^!\mathscr{F}, \eta : k_1^!i_1^!\mathscr{F} \xrightarrow{\sim} k_2^!i_2^!\mathscr{F})$$

is an equivalence. Indeed, consider the functor

$$\Phi : D(Z_1) \sqcup_{D(Y)} D(Z_2) \longrightarrow \overset{\text{l.lax}}{\lim}\left(D(U_1) \overset{i_1^*j_{1,*}}{\longrightarrow} D(Z_1)\right) \tag{3.3}$$
$$(\mathscr{F}_1, \mathscr{F}_2, \eta : k_1^!\mathscr{F}_1 \xrightarrow{\sim} k_2^!\mathscr{F}_2) \longmapsto (l_2^!\mathscr{F}_2, \mathscr{F}_1, \alpha : i_1^*j_{1,*}l_2^!\mathscr{F}_2 \longrightarrow \mathscr{F}_1[1]),$$

where α is obtained as follows. One applies the $(k_{1,*}, k_1^!)$ adjunction to the composite:

$$k_2^*l_{2,*}l_2^!\mathscr{F}_2 \xrightarrow{g_{\mathscr{F}_2}} k_2^!\mathscr{F}_2[1] \underset{\simeq}{\xrightarrow{\eta^{-1}}} k_1^!\mathscr{F}_1[1]$$

to obtain:

$$\alpha : i_1^*j_{1,*}l_2^! \simeq i_1^*i_{2,*}l_{2,*}l_2^!\mathscr{F}_2 \simeq k_{1,*}k_2^*l_{2,*}l_2^!\mathscr{F}_2 \longrightarrow \mathscr{F}_1[1],$$

where we used the base change with respect to (3.1) and the compatible isomorphisms $k_{1,!} \xrightarrow{\sim} k_{1,*}$, $i_{2,!} \xrightarrow{\sim} i_{2,*}$. It is straight-forward to check that Ψ and Φ are inverse to each other. $\square$

Proof of Proposition 3.1. Notice that since each $\imath_{\overline{p}}$ is fp-closed, Lemma 1.18 implies that $(\imath_{\overline{p}})_* : D(\mathscr{X}_{\overline{p}}) \hookrightarrow D(\mathscr{X})$ is a closed subcategory.

From §1.5.5 it is clear that we have equivalences:

$$\underset{p\in \mathsf{P}}{\operatorname{colim}} D(\mathscr{X}_{\overline{p}}) \xrightarrow{\sim} D(\mathscr{X}) \xrightarrow{\sim} \lim_{p\in\mathsf{P}} D(\mathscr{X}_{\overline{p}}).$$

We now check conditions (ii) and (iii) from Definition A.4. Notice that for any $p, q \in \mathsf{P}$ we have:

$$\mathscr{X}_{\overline{p}} \cap \mathscr{X}_{\overline{q}} = \cup_{r \leqslant p \text{ and } r \leqslant q} \mathscr{X}_{\overline{r}}. \tag{3.4}$$

The following

$$\begin{tikzcd} D(\mathscr{X}_{\overline{p}} \cap \mathscr{X}_{\overline{q}}) \arrow[r] \arrow[d] & D(\mathscr{X}_{\overline{p}}) \arrow[d, "\imath_{\overline{p},*}"] \\ D(\mathscr{X}_{\overline{q}}) \arrow[r, "i_{\overline{q},*}"] & D(\mathrm{Fl}) \end{tikzcd}$$

is a pullback square. Then (3.4) implies $D(\mathscr{X}_{\overline{p}} \cap \mathscr{X}_{\overline{q}}) \simeq D(\cup_{r \leqslant p \text{ and } r \leqslant q} \mathscr{X}_{\overline{r}})$. By iterating Lemma 3.2 we have that $\bigcup_{r \leqslant p \text{ and } r \leqslant q} D(\mathscr{X}_{\overline{r}}) \xrightarrow{\sim} D(\cup_{r \leqslant p \text{ and } r \leqslant q} \mathscr{X}_{\overline{r}})$. This gives condition (iii) from Definition A.4.

Now by base change, the following diagram commutes:

$$\begin{tikzcd} D(\mathscr{X}_{\overline{p} \cap \overline{q}}) \arrow[r, "(\imath_{\overline{p}\cap\overline{q},\overline{q}})_!"] & D(\mathscr{X}_{\overline{q}}) \\ D(\mathscr{X}_{\overline{p}}) \arrow[u, "(\imath_{\overline{p}\cap\overline{q},\overline{p}})^*"] \arrow[r, "(\imath_{\overline{p}})_!"'] & D(\mathscr{X}) \arrow[u, "(\imath_{\overline{q}})^*"'] \end{tikzcd},$$

thus we obtain condition (ii) from Definition A.4. □

3.2. **Affine flag variety.**

We apply the results of §3.1 to the affine flag variety.

3.2.1. *$\breve{W}$-stratification of $D(\mathrm{Fl})$.* We keep the notations of §2.3.1. Condition (i) of Proposition 3.1 is clear, whereas Condition (ii) follows from Remark 2.8. Thus, we have:

Lemma 3.3. *The functor:*

$$\breve{W} \longrightarrow \mathrm{Cl}_{D(\mathrm{Fl})}, \quad w \longmapsto (\imath_{\overline{w}})_* : D(\mathrm{Fl}_{\overline{w}}) \hookrightarrow D(\mathrm{Fl}). \tag{3.5}$$

gives a $\breve{W}$-closed stratification of $D(\mathrm{Fl})$.

In the rest of this section we spell out explicit the consequences of §A.3 and §A.4 for the stratification (3.5).

3.2.2. *Closed glueing.* We work out the details of the construction of §A.4.2 for (3.5).

For $w \in W$, we have:

$$\operatorname*{colim}_{w' \in \breve{W}_{\ngeqslant w}} D(\mathrm{Fl}_{\leqslant w'}) \simeq \lim_{w' \in (\breve{W}_{\ngeqslant w})^{\mathrm{op}}} D(\mathrm{Fl}_{\leqslant w'})$$

where the limit is computed in Pr^{L} with the connecting morphisms given by $*$-pullback. Let $\mathrm{Fl}_{\partial w} = \bigcup_{w' \ngeqslant w} \mathrm{Fl}_{w'}$ then $\{\mathrm{Fl}_{\leqslant w'} \to \mathrm{Fl}_{\partial w}\}_{w' \ngeqslant w}$ is a Zariski cover, thus by descent, we obtain that

$$\operatorname*{colim}_{w' \in W_{\ngeqslant w}} D(\mathrm{Fl}_{\leqslant w'}) = D(\mathrm{Fl}_{\partial w}).$$

Hence, the recollement diagram:

$$\begin{tikzcd}
D(\mathrm{Fl})_w \arrow[r, hook, bend left, "\pi_w^{\mathrm{L}}"] \arrow[r, hook, bend right, "\pi_w^{\mathrm{R}}"'] & D(\mathrm{Fl}_{\leqslant w}) \arrow[l, "\pi_w"] \arrow[r, bend left, "(i_{\partial w})^*"] \arrow[r, bend right, "(i_{\partial w})^!"'] & D(\mathrm{Fl}_{\partial w}), \arrow[l, hook', "(i_{\partial w})_*"]
\end{tikzcd}$$

defining the strata $D(\mathrm{Fl})_w$, is equivalent to:

$$\begin{tikzcd}
D(\mathrm{Fl}_w) \arrow[r, hook, bend left, "(j_w)_!"] \arrow[r, hook, bend right, "(j_w)_*"'] & D(\mathrm{Fl}_{\leqslant w}) \arrow[l, "j_w^!"] \arrow[r, bend left, "(i_{\partial w})^*"] \arrow[r, bend right, "(i_{\partial w})^!"'] & D(\mathrm{Fl}_{\partial w}). \arrow[l, hook', "(i_{\partial w})_*"]
\end{tikzcd}$$

For each $w \in W$ we have the composite adjunctions:

$$\Psi^w : D(\mathrm{Fl}_w) \underset{(j_w)^!}{\overset{(j_w)_!}{\rightleftarrows}} D(\mathrm{Fl}_{\leqslant w}) \underset{(i_{\leqslant w})^!}{\overset{(i_{\leqslant w})_*}{\rightleftarrows}} D(\mathrm{Fl}) : \nu_w \;\;,$$

since $(i_{\leqslant w})_! \xrightarrow{\sim} (i_{\leqslant w})_*$, we have $\Psi^w = (i_w)_!$ and $\nu_w = (i_w)^!$.

Notice that given $w_1 \leqslant w_2 \leqslant w_3$ we obtain a diagram:

$$\begin{tikzcd}
& D(\mathrm{Fl}_{w_2}) \arrow[dr, "(j_{w_1})^! \circ (j_{w_2})_!"] \arrow[d, phantom, "\Downarrow"] & \\
D(\mathrm{Fl}_{w_3}) \arrow[ur, "(j_{w_2})^! \circ (j_{w_3})_!"] \arrow[rr, "(j_{w_1})^! \circ (j_{w_3})_!"'] & {} & D(\mathrm{Fl}_{w_1})
\end{tikzcd}$$

Here the double arrow means there is a (*not necessarily invertible*) 2-morphism $(j_{w_1})^! \circ (j_{w_2})_! \circ (j_{w_2})^! \circ (j_{w_3})_! \to (j_{w_1})^! \circ (j_{w_3})_!$.

More generally, the functors $j^!_{w'} \circ (j_w)_!$ assemble into a right-lax $\breve{W}^{\mathrm{op}}$-module(see [3, §A.1] for a precise definition):

$$\breve{W}^{\mathrm{op}} \xrightarrow[\mathrm{r.lax}]{D(\mathrm{Fl}_-)} \mathrm{Pr}^{\mathrm{L}}.$$

From Theorem A.9 we obtain:

$$D(\mathrm{Fl}) \xrightarrow{\sim} \lim^{\mathrm{l.lax}}_{\mathrm{r.lax}.W^{\mathrm{op}}} D(\mathrm{Fl}_w), \tag{3.6}$$

which can be more concretely computed as the strict colimit:

$$\operatorname*{colim}_{\mathrm{sd}(W^{\mathrm{op}})^{\mathrm{op}}} D(\mathrm{Fl}_w) \xrightarrow{\sim} D(\mathrm{Fl}). \tag{3.7}$$

3.2.3. *Open decomposition.* For each $w \in \breve{W}$, let $\mathrm{Fl}_{\geqslant w} := \bigsqcup_{w' \geqslant w} \mathrm{Fl}_{w'} \simeq \bigsqcup_{w' \nleqslant w} \mathrm{Fl}_{\leqslant w'}$. Note that $\mathrm{Fl}_{\geqslant w} = (\mathrm{Fl} \backslash \mathrm{Fl}_{\leqslant w}) \cup \mathrm{Fl}_w$, $\mathrm{Fl}_w \hookrightarrow \mathrm{Fl}$ is a qcqs open since $(\mathrm{Fl} \backslash \mathrm{Fl}_{\leqslant w}) \hookrightarrow \mathrm{Fl}$ is a qcqs open as the complement of a fp-closed morphism. Thus the embedding $i_{\geqslant w} : \mathrm{Fl}_{\geqslant w} \hookrightarrow \mathrm{Fl}$ is a qc qs as the union of two such. The functor $i_{\geqslant w,!} : D(\mathrm{Fl}_{\geqslant w}) \hookrightarrow D(\mathrm{Fl})$ exhibits $D(\mathrm{Fl}_w)$ as an open subcategory, i.e. we have adjunctions $(i_{\geqslant w,!}, i^!_{\geqslant w})$ and $(i^!_{\geqslant w}, i_{\geqslant w,*})$.

The same argument as in the proof of Lemma 3.3 gives the following result:

Lemma 3.4. *The functor:*

$$
\breve{W}^{\mathrm{op}} \longrightarrow \mathrm{Op}_{D(\mathrm{Fl})}; \qquad w \longmapsto \begin{tikzcd} D(\mathrm{Fl}_{\geqslant w}) \arrow[r, bend left, "(i_{\geqslant w})_!"] \arrow[r, bend right, "(i_{\geqslant w})_*"'] & D(\mathrm{Fl}) \arrow[l, dashed, "i^!_{\geqslant w}"'] \end{tikzcd} \tag{3.8}
$$

determines a $\breve{W}^{\mathrm{op}}$-stratification of $D(\mathrm{Fl})$.

We could try to perform the open glueing (see §A.4.1) associated with this data, however, since $\breve{W}^{\mathrm{op}}$ is *not* down-finite, the data obtained would not be equivalent to the open stratification, i.e. it would not recover the category $D(\mathrm{Fl})$ (see Figure 1).

3.2.4. *Reflected glueing.* In this section we follow the procedure of constructing reflected open glueing diagrams as described in [3, §1.10]. Instead of using the open glueing functors (A.4), i.e. $*$-pushforward and pullback, we can consider the *reflected geometric localization diagrams*:

$$
\lambda^w : D(\mathrm{Fl}_w) \underset{(k_w)^!}{\overset{(k_w)_!}{\rightleftarrows}} D(U_w) \underset{(i_{\geqslant w})^!}{\overset{(i_{\geqslant w})_!}{\rightleftarrows}} D(\mathrm{Fl}) : \Psi_w \ . \tag{3.9}
$$

In this case these functors are simply given by $\lambda^w = (\imath_w)_!$ and $\Psi_w = (\imath_w)^!$. Thus, for $w \to w'$ in $\breve{W}^{\mathrm{op}}$, i.e. $w' \leqslant w$, the *reflected glueing functors* are:

$$
\breve{\Gamma}^w_{w'} : D(\mathrm{Fl}_w) \overset{(\imath_w)_!}{\hookrightarrow} D(\mathrm{Fl}) \overset{(\imath_{w'})^!}{\hookrightarrow} D(\mathrm{Fl}_{w'}).
$$

As before, these assemble into a *reflected glueing diagram*:

$$
\breve{W}^{\mathrm{op}} \xrightarrow[\mathrm{r.lax}]{\breve{\mathring{\mathscr{G}}}(D(\mathrm{Fl}))} \mathrm{Pr}^{\mathrm{L}} \ .
$$

By [3, Theorem F], we have an equivalence:

$$
\operatorname*{colim}_{\mathsf{sd}(\breve{W}^{\mathrm{op}})^{\mathrm{op}}} \breve{\mathring{\mathscr{G}}}(D(\mathrm{Fl})) \xrightarrow{\sim} D(\mathrm{Fl}). \tag{3.10}
$$

Moreover, Remark A.10 implies that (3.10) coincindes with the glueing given by (3.7).

3.3. **Decomposition of $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$.** Recall from §2.3 that LG is an ind-placid scheme. We have an action of LG on itself by θ-conjugation. We let $\frac{LG}{\mathrm{Ad}_\theta(LG)}$ denote the (étale) quotient stack, i.e. the étale sheafification of the prestack quotient $\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)_{\mathrm{PStk}}$. We claim that $\frac{LG}{\mathrm{Ad}_\theta(LG)}$ is a sifted-placid stack. Indeed, since the action $\mathrm{Ad}^{\mathrm{Iw}}_\theta : \mathrm{Iw} \times LG \to LG$ factors through $LG_{\overline{\mathcal{O}}}$ for some Newton point, hence $\mathrm{Ad}^{\mathrm{Iw}}_\theta$ is ind-fp proper and by Remark 1.13 (4) we obtain that $\frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})}$ is a sifted-placid stack. Since the canonical projection

$$
\pi : \frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \longrightarrow \frac{LG}{\mathrm{Ad}_\theta(LG)}
$$

is surjective ind-fp proper, we obtain that $\frac{LG}{\mathrm{Ad}_\theta(LG)}$ is a sifted-placid stack as well.

For every straight θ-conjugacy class $\mathcal{O}$ we denote by

$$
j_{\mathcal{O}} : \frac{LG_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)} \hookrightarrow \frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)} \quad \text{and} \quad i_{\overline{\mathcal{O}}} : \frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)} \hookrightarrow \frac{LG}{\mathrm{Ad}_\theta(LG)}
$$

the inclusions of the quotients of (2.3). We claim that $j_{\mathcal{O}}$ is qc qs and that $i_{\overline{\mathcal{O}}}$ is fp-closed. Indeed, since the notion of fp-closed embedding is local for the étale topology, it is enough to check that the morphisms

$$\left(\frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)}\right)_{\mathrm{PStk}} \hookrightarrow \left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)_{\mathrm{PStk}}$$

at the level of prestack quotients is finitely presented, which directly follows from Theorem 2.7. The argument for $j_{\mathcal{O}}$ is similar. We pose $i_{\mathcal{O}} := i_{\overline{\mathcal{O}}} \circ j_{\mathcal{O}}$.

Notice that $D\left(\frac{\breve{G}}{\mathrm{Ad}_\theta(\breve{G})}\right)$ is very far from the product $\prod_{\mathcal{O}} D\left(\frac{\breve{G}_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(\breve{G})}\right)$, as pushforward of sheaves coming from different Newton strata can interact. Roughly speaking, the best description of this category we can hope for is a form of semi-orthogonal decomposition.

Theorem 3.5. *The collection of functors:*

$$(i_{\overline{\mathcal{O}}})_* : D\left(\frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)}\right) \longrightarrow D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right) \tag{3.11}$$

determines a closed $\breve{W} /\!\!/_\theta \breve{W}$-stratification of $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$. Moreover, the $\mathcal{O}$-strata of the corresponding semi-orthogonal decomposition is given by:

$$D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)_{\mathcal{O}} \simeq D\left(\frac{LG_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right)$$

and closed glueing functors corresponding to $\mathcal{O} \preccurlyeq \mathcal{O}'$ are given by $(i_{\mathcal{O}'})^! \circ (i_{\mathcal{O}})_!$.

Remark 3.6. We call this decomposition the *Newton decomposition* of $D\left(\frac{\breve{G}}{\mathrm{Ad}_\theta(\breve{G})}\right)$.

Proof. By Theorem 2.7, the collection $\{LG_{\mathcal{O}}\}_{\mathcal{O}\in \breve{W}/\!\!/_\theta \breve{W}}$ satisfy the conditions of Proposition 3.1. Thus, the assignment (3.11) gives a $\breve{W} /\!\!/_\theta \breve{W}$-indexed closed stratification.

We now compute the $\mathcal{O}$-strata. Let $\frac{LG_{\partial\mathcal{O}}}{\mathrm{Ad}_\theta LG} := \bigcup_{\mathcal{O}' \not\preccurlyeq \mathcal{O}} \frac{LG_{\overline{\mathcal{O}'}}}{\mathrm{Ad}_\theta(LG)}$ and $i_{\partial\mathcal{O}} : \frac{LG_{\partial\mathcal{O}}}{\mathrm{Ad}_\theta LG} \hookrightarrow \frac{LG}{\mathrm{Ad}_\theta LG}$ the associated fp-closed embedding. One has an equivalence:

$$\operatorname*{colim}_{\mathcal{O}'\not\preccurlyeq\mathcal{O}} D\left(\frac{LG_{\overline{\mathcal{O}'}}}{\mathrm{Ad}_\theta(LG)}\right) \xrightarrow{\sim} D\left(\frac{LG_{\partial\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right).$$

Thus, the recollement diagram defining $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)_{\mathcal{O}}$ becomes:

$$\begin{tikzcd}
D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)_{\mathcal{O}} \arrow[r, hook, bend left, "\pi^{\mathrm{L}}_{\mathcal{O}}"] \arrow[r, hook, bend right, "\pi^{\mathrm{R}}_{\mathcal{O}}"'] & D\left(\frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[l, "\pi_{\mathcal{O}}"'] \arrow[r, bend left, "(i_{\partial\mathcal{O}})^*"] \arrow[r, bend right, "(i_{\partial\mathcal{O}})^!"'] & D\left(\frac{LG_{\partial\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[l, hook', "(i_{\partial\mathcal{O}})_*"']
\end{tikzcd} \tag{3.12}$$

By Proposition A.1 (2) we have that $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)_{\mathcal{O}}$ is equivalent to:

$$\left\{ \mathscr{F} \in D\left(\frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)}\right) \mid \mathrm{Map}(\mathscr{F}, (i_{\partial\mathcal{O}})_*(\mathscr{G})) = 0, \text{ for all } \mathscr{G} \in D\left(\frac{LG_{\partial\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right)\right\}.$$

Since $i_{\partial\mathcal{O}}$ is fp-closed, by adjunction we obtain that $(i_{\partial\mathcal{O}})^*(\mathscr{F}) \simeq 0$. Thus, (1.31) implies that $\mathscr{F} \simeq (j_{\mathcal{O}})_!(\mathscr{F}')$ for some $\mathscr{F}' \in D\left(\frac{LG_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right)$.

This shows that the recollement diagram (3.12) is equivalent to:

$$\begin{tikzcd} D\left(\frac{LG_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[r, bend left, "(j_{\mathcal{O}})_!"] \arrow[r, bend right, "(j_{\mathcal{O}})_*"'] & D\left(\frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[l, "(j_{\mathcal{O}})^!"'] \arrow[r, bend left, "(i_{\partial\mathcal{O}})^*"] \arrow[r, bend right, "(i_{\partial\mathcal{O}})^!"'] & D\left(\frac{LG_{\partial\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[l, hook', "(i_{\partial\mathcal{O}})_*"'] \end{tikzcd}.$$

One can then directly check that the glueing functors are as claimed. This finishes the proof. ☐

3.4. **Further comments.** We point out that as a consequence of Theorem 3.5 we have:

$$\operatorname*{colim}_{\mathcal{O}\in\breve{W}/\!\!/_\theta\breve{W}} D\left(\frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)}\right) \xrightarrow{\sim} D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right).$$

Concretely, it means that any sheaf $\mathscr{F} \in D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$ is $(i_{\overline{\mathcal{O}}})_*(\mathscr{F}_{\overline{\mathcal{O}}})$ for some $\mathscr{F}_{\overline{\mathcal{O}}} \in D(\frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)})$. Moreover, compact objects in $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$ are given by direct images of $D_{\mathrm{c}}(\frac{LG_{\overline{\mathcal{O}}}}{\mathrm{Ad}_\theta(LG)})$ for some $\mathcal{O}$.

Alternatively, the theory of stratified stable ∞-categories gives a different description of sheaves on $\frac{LG}{\mathrm{Ad}_\theta(LG)}$ in terms of their values on the locally closed disjoint strata of LG. Indeed, by applying Theorem A.9 to Theorem 3.5 we obtain the following:

Corollary 3.7. *The category $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$ can be recovered from its locally closed pieces as the following strict colimit:*

$$\operatorname*{colim}_{\mathsf{sd}(\breve{W}/\!\!/_\theta\breve{W})} D\left(\frac{LG_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \xrightarrow{\sim} D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right).$$

Let's unwind this statement. First, for a poset P one has:

$$\mathsf{sd}(\mathsf{P}) := \{\varphi : [n] \longrightarrow \mathsf{P} \mid \varphi \text{ injective morphism of posets}\}.$$

Notice that as a subset of the power set of P, $\mathsf{sd}(\mathsf{P})$ is itself a poset. Given $\mathscr{F} \in D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$, let $g(\mathscr{F}) \in \lim^{\mathrm{l.lax}}_{\mathrm{r.lax},\breve{W}/\!\!/_\theta\breve{W}} \mathscr{G}(\mathscr{F})$ denote the data of[7]

$$\left(\{i_{\mathcal{O}}^!(\mathscr{F}) \in D(\frac{LG_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)})\}_{\mathcal{O}\in\breve{W}/\!\!/_\theta\breve{W}}, \{\alpha_{\mathcal{O}_1,\mathcal{O}_2} : \Gamma^{\mathcal{O}_1}_{\mathcal{O}_2} \circ i^!_{\mathcal{O}_1}(\mathscr{F}) \longrightarrow i^!_{\mathcal{O}_2}(\mathscr{F})\}_{\mathcal{O}_1 \not\geq \mathcal{O}_2}\right),$$

where $\alpha_{\mathcal{O}_1,\mathcal{O}_2}$ are induced by the canonical counit morphisms, satisfying certain compatibilities. Then one has:

$$\operatorname*{colim}_{\mathsf{sd}(\mathcal{O}\in\breve{W}/\!\!/_\theta\breve{W})} i^!_{\mathcal{O}}(\mathscr{F}) \xrightarrow{\sim} \mathscr{F}.$$

In fact, any collection of sheaves and morphisms:

$$\left(\{\mathscr{F}_{\mathcal{O}} \in D(\frac{LG_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)})\}_{\mathcal{O}\in\breve{W}/\!\!/_\theta\breve{W}}, \{\alpha_{\mathcal{O}_1,\mathcal{O}_2} : \Gamma^{\mathcal{O}_1}_{\mathcal{O}_2}(\mathscr{F}_{\mathcal{O}_1}) \longrightarrow \mathscr{F}_{\mathcal{O}_2}\}_{\mathcal{O}_1 \not\geq \mathcal{O}_2}\right)$$

satisfying conditions analogous to those specified in Example [3, Example A.5.3. (2)] determines a sheaf $\mathscr{F} := \operatorname{colim}_{\mathsf{sd}(\mathcal{O}\in\breve{W}/\!\!/_\theta\breve{W})} \mathscr{F}_{\mathcal{O}}$ in $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$.

[7]Notice that for $\mathcal{O}_1 \not\geq \mathcal{O}_2$ the functor $\Gamma^{\mathcal{O}_1}_{\mathcal{O}_2}$ vanishes, so by Remark A.8 we don't need $\alpha_{\mathcal{O}_1,\mathcal{O}_2}$ in this case.

Furthermore, let $\mathscr{E} \in D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$ denote another sheaf, then one has an equivalence (cf. [3, Theorem A (4)]):

$$\mathrm{Map}_{D(\frac{LG}{\mathrm{Ad}_\theta(LG)})}(\mathscr{F},\mathscr{E}) = \lim_{([n]\xrightarrow{\varphi} W/\!/_\theta \check{W})\in \mathsf{sd}(\check{W}/\!/_\theta\check{W})^{\mathrm{op}}} \left(\mathrm{Map}_{D(\frac{LG_{\varphi(n)}}{\mathrm{Ad}_\theta(LG)})}(\Gamma^\varphi \circ i^!_{\varphi(0)}(\mathscr{F}), i^!_{\varphi(0)}(\mathscr{E}))\right)$$

where $\Gamma^\varphi := i^!_{\varphi(n)} \circ (i_{\varphi(n-1)})_! \circ \cdots \circ i^!_{\varphi(1)} \circ (i_{\varphi(0)})_!$.

4. Categorical cocenter of affine Hecke category

4.1. **Categorical cocenter.** We start by recalling some general categorical constructions. Given a monoidal presentable stable ∞-category $(\mathcal{C},\star)$ the *categorical cocenter* $\mathrm{Tr}(\mathcal{C})$ *of* $\mathcal{C}$ is defined as $\mathrm{Tr}(\mathcal{C}) := \mathcal{C}\otimes_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}\mathcal{C}$, where $\mathcal{C}^{\mathrm{rev}}$ denotes $\mathcal{C}$ with the reversed monoidal structure. The tensor product is taken in the category Lincat_E. It receives a universal map $\mathrm{tr}: \mathcal{C}\to\mathrm{Tr}(\mathcal{C})$. In the literature $\mathrm{Tr}(\mathcal{C})$ is also referred to as the *categorical trace* or *categorical Hochschild homology* of $\mathcal{C}$.

Now we discuss the twisted cocenter. Let $\Phi:\mathcal{C}\to\mathcal{C}$ be a (right-lax) monoidal endofunctor. We denote by $\mathcal{C}$-mod the ∞-category of left $\mathcal{C}$-modules in presentable stable ∞-categories. One obtains $\Phi_*:\mathcal{C}\text{-mod}\to\mathcal{C}\text{-mod}$ which is represented by ${}_\Phi\mathcal{C}\in\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}$-mod, i.e. ${}_\Phi\mathcal{C}=\mathcal{C}$ with the left action of $\mathcal{C}$ via Φ and the right action as usual. We define (following [9, §3.7.1] [20, §7.3.4]) the Φ-*twisted categorical cocenter of* $\mathcal{C}$ as:

$$\mathrm{Tr}(\Phi,\mathcal{C}) := \mathcal{C}\underset{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}{\otimes}{}_\Phi\mathcal{C}. \tag{4.1}$$

We denote by $\mathrm{tr}:\mathcal{C}\to\mathrm{Tr}(\Phi,\mathcal{C})$ the canonical map sendsing $c\in\mathcal{C}$ to the image of $1_{\mathcal{C}}\otimes c$ in $\mathrm{Tr}(\Phi,\mathcal{C})$.

Let Δ denote the simplex category, i.e. the objects are non-empty totally ordered finite sets $[n] := \{0<\cdots<n\}$ for $n\geqslant 0$ and the morphisms are order-preserving functions. To concretely calculate (4.1) we consider the bar resolution of $\mathcal{C}$ as a right $\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}$-module, given by:

$$\mathrm{Bar}(\mathcal{C})_\bullet : \Delta^{\mathrm{op}}\longrightarrow \mathrm{Lincat}_E, \quad \mathrm{Bar}(\mathcal{C})_n := \mathcal{C}^{\otimes n}\otimes\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}\simeq\mathcal{C}^{\otimes(n+2)}$$

with morphisms induced by the monoidal structure. By tensoring with $(-)\underset{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}{\otimes}{}_\Phi\mathcal{C}$ one obtains:

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}}\mathrm{Bar}(\mathcal{C})_\bullet\underset{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}{\otimes}{}_\Phi\mathcal{C}\xrightarrow{\sim}\mathrm{Tr}(\Phi,\mathcal{C}).$$

One can simplify the above resolution by level-wise using the isomorphism:

$$\begin{gathered}\mathcal{C}^{\otimes(n+2)}\underset{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}{\otimes}{}_\Phi\mathcal{C}\xrightarrow{\sim}\mathcal{C}^{\otimes n}\otimes{}_\Phi\mathcal{C}\\ (a_0,a_1,\cdots,a_n,a_{n+1},b)\longmapsto(a_1,\cdots,a_n,\Phi(a_{n+1})ba_0).\end{gathered}$$

The resulting simplicial object in the ∞-category of presentable stable ∞-categories is the (twisted) *cyclic bar construction*:

$$\mathcal{C}_{\Phi,\bullet}:\Delta^{\mathrm{op}}\longrightarrow\mathrm{Lincat}_E,\quad [n]\longmapsto\mathcal{C}^{\otimes n}\otimes{}_\Phi\mathcal{C}, \tag{4.2}$$

where the face morphisms $\mathcal{C}^{\otimes n+1} \otimes {}_{\Phi}\mathcal{C} \to \mathcal{C}^{\otimes n} \otimes {}_{\Phi}\mathcal{C}$ are given by:

$$\begin{aligned} d_{0,n} &:= \mathrm{id}_{\mathcal{C}^{\otimes n}} \otimes \Phi(-) \star (-) \\ d_{i,n} &:= \mathrm{id}_{\mathcal{C}^{(i-1)}} \otimes (-) \star (-) \otimes \mathrm{id}_{\mathcal{C}^{\otimes (n-i)} \otimes_{\Phi} \mathcal{C}} \ \text{ for } 0 < i < n+1, \\ d_{n+1,n} &:= \mathrm{id}_{\mathcal{C}^{\otimes n}} \otimes {}_{\Phi}(-) \star (-)_1, \end{aligned}$$

where ${}_{\Phi}(-) \star (-)_1$ means we apply the tensor product with ${}_{\Phi}\mathcal{C}$ as the left factor and the first copy of $\mathcal{C}$ as the right factor. Thus, we have (cf. [28, Theorem 5.5.3.11 and Remark 5.5.3.13]):

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathcal{C}_{\Phi,\bullet} \xrightarrow{\sim} \mathrm{Tr}(\Phi, \mathcal{C}).$$

Later we will also need the degeneracy morphisms:

$$c_{i,n} := \mathrm{id}_{\mathcal{C}^{\otimes i}} \otimes 1_{\mathcal{C}} \otimes \mathrm{id}_{\mathcal{C}^{\otimes (n-i)} \otimes_{\Phi} \mathcal{C}} : \mathcal{C}^{\otimes n} \otimes {}_{\Phi}\mathcal{C} \longrightarrow \mathcal{C}^{\otimes (n+1)} \otimes {}_{\Phi}\mathcal{C},$$

where $1_{\mathcal{C}} \in \mathcal{C}$ is the unit of the monoidal structure of $\mathcal{C}$, for $0 \leqslant i \leqslant n$.

4.2. **Beck–Chevalley criterion.** To have a more concrete understanding of the cocenter we would like to realize it as a full subcategory of a more accessible category. For that, we recall an abstract result of Lurie that allows us to do that. Let Δ_+ denote the augmented simplex category, i.e. the objects are (possibly) empty totally ordered finite sets with order-preserving functions as morphisms. By convention, $[-1]$ denotes the empty set. Given an augmented simplicial object $\mathcal{D}^+_\bullet : \Delta^{\mathrm{op}}_+ \to \mathrm{Lincat}_E$, one says that $\mathcal{D}^+_\bullet$ satisfies the *Beck–Chevally condition* if for every $n, m \geqslant -1$ and every morphism $\alpha : [n] \to [m]$ in Δ_+, inducing $\alpha_+ : [n+1] = [0] \star [n] \to [0] \star [m] = [m+1]$, the induced diagram:

$$\begin{tikzcd} \mathcal{D}^+_{m+1} \arrow[r, "d_{0,m}"] \arrow[d, "\mathcal{D}^+_{\alpha_+}"'] & \mathcal{D}^+_m \arrow[d, "\mathcal{D}^+_\alpha"] \\ \mathcal{D}^+_{n+1} \arrow[r, "d_{0,n}"'] & \mathcal{D}^+_n \end{tikzcd}$$

is vertically right adjointable (see §1.1 (g)). The following result is proved in [28, Corollary 4.7.5.3] and [21, Proposition 2.3.3]:

Proposition 4.1. *Given a simplicial object $\mathcal{D}_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Lincat}_E$ and an extension $\mathcal{D}^+_\bullet : \Delta^{\mathrm{op}}_+ \to \mathrm{Lincat}_E$, i.e. $\mathcal{D}^+_\bullet|_{\Delta^{\mathrm{op}}} = \mathcal{D}_\bullet$, of $\mathcal{D}_\bullet$ to an augmented simplicial object such that:*

(i) the unique face map $\mathcal{D}^+_0 \to \mathcal{D}^+_{-1}$ admits a right adjoint,

(ii) $\mathcal{D}^+_\bullet$ satisfies the Beck–Chevalley condition.

Then the canonical morphism:

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathcal{D}_\bullet \longrightarrow \mathcal{D}^+_{-1}$$

is fully faithful and admits a right adjoint. Moreover, the category $\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathcal{D}_\bullet$ is generated under colimits by the essential image of $\mathcal{D}^+_0 \to \mathcal{D}^+_{-1}$.

In Proposition 4.1 the last statement follows by applying [28, Proposition 4.7.3.14] to the adjunction $F : \mathcal{D}^+_0 \rightleftarrows \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathcal{D}_\bullet : G$, where $F : \mathscr{C}^+_0 \to \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathcal{D}_\bullet$ is the canonical morphism to the colimit and G its right adjoint, whose existence follows from the proof of Proposition 4.1 as in [21, Proposition 2.3.3].

More concretely, here is what one needs to check the Beck–Chevaley condition. Firstly, notice that any morphism in the category Δ is a composition of two types of morphisms:

degeneracy morphisms and face morphisms. Thus, by functoriality of passing to right adjoints, we only need to check that for every $n \geqslant 0$ and $0 \leqslant i \leqslant n+1$ the induced diagrams:

$$\begin{tikzcd} \mathcal{D}_{n+1} \arrow[r, "d_{0,n}"] \arrow[d, "c_{i+1,n+2}"'] & \mathcal{D}_n \arrow[d, "c_{i,n+1}"] \\ \mathcal{D}_{n+2} \arrow[r, "d_{0,n+1}"'] & \mathcal{D}_{n+1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{D}_{n+2} \arrow[r, "d_{0,n+1}"] \arrow[d, "d_{i+1,n+1}"'] & \mathcal{D}_{n+1} \arrow[d, "d_{i,n}"] \\ \mathcal{D}_{n+1} \arrow[r, "d_{0,n}"'] & \mathcal{D}_n \end{tikzcd}$$

are vertically right adjointable.

Secondly, notice that an extension of $\mathcal{D}_\bullet$ to an augmented simplicial object $\mathcal{D}^+_\bullet : \Delta^{\mathrm{op}}_+ \to \mathrm{Lincat}_E$, is the data of a presentable stable ∞-category $\mathcal{D}^+_{-1}$, a functor $d_{0,-1} : \mathcal{D}_0 \to \mathcal{D}^+_{-1}$ and an isomorphism

$$\alpha : d_{0,-1} \circ d_{0,0} \xrightarrow{\simeq} d_{0,-1} \circ d_{1,0}.$$

These are required to satisfy higher compatibilities with the isomorphisms determining $\mathcal{D}_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Lincat}_E$, e.g. the following diagram of isomorphisms commutes:

$$\begin{tikzcd} d_{0,-1}d_{0,1}d_{0,2} \arrow[r, "\beta_2"] \arrow[d, "\alpha"'] & d_{0,-1}d_{0,1}d_{1,2} \arrow[r, "\alpha"] & d_{0,-1}d_{1,1}d_{1,2} \arrow[d, "\beta_0"] \\ d_{0,-1}d_{1,1}d_{0,2} \arrow[r, "\beta_1"] & d_{0,-1}d_{0,1}d_{2,2} \arrow[r, "\alpha"] & d_{0,-1}d_{1,1}d_{2,2} \end{tikzcd},$$

and so on for higher compositions. We emphasize that there is no general way to construct $\mathcal{D}^+_{-1}$ and $d_{0,-1}$.

Finally, the Beck–Chevalley condition for $\mathcal{C}_{-1}$ requires that the diagram:

$$\begin{tikzcd} \mathcal{D}_1 \arrow[r, "d_{0,0}"] \arrow[d, "d_{1,0}"'] & \mathcal{D}_0 \arrow[d, "d_{0,-1}"] \\ \mathcal{D}_0 \arrow[r, "d_{0,-1}"] & \mathcal{D}^+_{-1} \end{tikzcd}$$

is vertically right adjointable. Notice that there is no diagram involving degeneracy maps and the category $\mathcal{D}^+_{-1}$, since there are no morphisms from $[-1]$ to $[0]$ in Δ_+.

4.3. **Affine Hecke category.** Let $X = \mathrm{Iw}\backslash LG/\mathrm{Iw}$, in this section we focus on the affine Hecke category $\mathcal{H} = D(X)$. Notice that for each $w \in \breve{W}$ by Remark 1.13 (2) one has a placid atlas:

$$h_w : LG_{\leqslant w} \longrightarrow \mathrm{Iw}\backslash LG_{\leqslant w}/\mathrm{Iw}.$$

In fact, since one has a pro-unipotent radical placid group subscheme $\mathrm{Iw}^{\mathrm{u}} \hookrightarrow \mathrm{Iw}$, whose quotient $\mathrm{Iw}/\mathrm{Iw}^{\mathrm{u}}$ is of finite presentation, one concludes that h_w is essentially cohomologically smooth, so $\mathrm{Iw}\backslash LG_{\leqslant w}/\mathrm{Iw}$ is very placid. Thus, $\operatorname{colim}_{w\in\breve{W}} \mathrm{Iw}\backslash LG_{\leqslant w}/\mathrm{Iw} \xrightarrow{\sim} X$ is an ind-very placid stack.

We first explain the monoidal structure of $\mathcal{H}$. The placid group scheme $\mathrm{Iw}\times\mathrm{Iw}$ acts on $\mathrm{Iw}\backslash LG \times LG/\mathrm{Iw}$ via $((h_1,h_2),[g_1,g_2]) \mapsto [g_1h_1^{-1}, h_2g_2]$. Let $Z = \mathrm{Iw}\backslash LG \times^{\mathrm{Iw}} LG/\mathrm{Iw}$ be the quotient stack of $\mathrm{Iw}\backslash LG \times LG/\mathrm{Iw}$ with respect to the diagonal Iw-action. We have the following diagram:

$$X \times X \xleftarrow{a} Z \xrightarrow{b} X,$$

where a is induced by the quotient with respect to Iw sitting diagonally in $\mathrm{Iw}\times\mathrm{Iw}$ and b is induced by the multiplication map.

The monoidal structure is given by the composite:

$$\star : D(X) \otimes D(X) \xrightarrow{\boxtimes} D(X \times X) \xrightarrow{b_* \circ a^!} D(X), \tag{4.3}$$

where $\boxtimes : D(X) \otimes D(X) \to D(X \times X)$ is the external tensor product encoded in the lax monoidal structure of (1.27). In general, the external tensor product functor is only fully faithful, see [20, Lemma 10.85, (10.48)]. However, for the affine Hecke category we have:

Lemma 4.2. *[20, §3.2] The external tensor product $\boxtimes : D(X)\otimes D(X) \to D(X\times X)$ induces an equivalence of presentable stable ∞-categories. Thus, $\mathcal{H}_n \simeq D(X^{(n+1)})$ for $n \geqslant 0$.*

The automorphism θ of LG induces an automorphism of X that we still denote as θ. Let $\Phi = \theta^! : \mathcal{H} \to \mathcal{H}$ denote the induced monoidal functor on $\mathcal{H}$.We will compute the Φ-twisted cocenter of $\mathcal{H}$ using Proposition 4.1.

The correspondence $\mathrm{Iw}\backslash LG/\mathrm{Iw} \xleftarrow{p} \frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \xrightarrow{q} \frac{LG}{\mathrm{Ad}_\theta(LG)}$ gives a functor:

$$CH := q_* \circ p^! : \mathcal{H} \longrightarrow D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right). \tag{4.4}$$

By definition of the trace, one has a factorization:

$$CH : \mathcal{H} \xrightarrow{\mathrm{tr}} \mathrm{Tr}(\Phi, \mathcal{H}) \xrightarrow{F} D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right).$$

We now state the main result of this section.

Theorem 4.3. *(1) The canonical functor:*

$$F : \mathrm{Tr}(\Theta, \mathcal{H}) \longrightarrow D\left(\frac{\breve{G}}{\mathrm{Ad}_\theta(\breve{G})}\right) \tag{4.5}$$

is fully faithful and admits a right adjoint. Moreover, $\mathrm{Tr}(\Theta,\mathcal{H})$ is generated under colimits by the essential image of CH.

(2) For each $\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}$, let $\mathrm{Tr}(\Theta,\mathcal{H})_{\succeq\mathcal{O}}$ be the subcategory of $\mathrm{Tr}(\Theta,\mathcal{H})$ spanned by sheaves $\mathscr{F}$ such that $(i_{\overline{\mathcal{O}'}})^*(\mathscr{F}) = 0$[8] for all $\mathcal{O}' \not\succeq \mathcal{O}$. The assignment

$$W /\!\!/_\theta \breve{W} \longrightarrow \mathrm{Op}_{\mathrm{Tr}(\Theta,\mathcal{H})}, \qquad \mathcal{O} \longmapsto \mathrm{Tr}(\Theta,\mathcal{H})_{\succeq\mathcal{O}}$$

determines an open $\breve{W} /\!\!/_\theta \breve{W}$-stratification of $\mathrm{Tr}(\Theta,\mathcal{H})$.

(3) The stratum $\mathrm{Tr}(\Theta,\mathcal{H})_{\mathcal{O}}$ of the semi-orthogonal decomposition consists of $\mathscr{F} \in \mathrm{Tr}(\Theta,\mathcal{H})$, such that $\mathscr{F} \simeq (i_{\mathcal{O}})_!(\mathscr{G})$ for some $\mathscr{G} \in D\left(\frac{LG_{\mathcal{O}}}{LG}\right)$. More concretely, $\mathrm{Tr}(\Theta,\mathcal{H})_{\mathcal{O}}$ is generated under colimits by the essential image of $q_{w,*} \circ p_w^! : D(\mathrm{Iw}\backslash LG_w/\mathrm{Iw}) \to D\left(\frac{LG}{LG}\right)$ where $w \in \breve{W}$ such that $LG_w \subset LG_{\mathcal{O}}$ and $\mathrm{Iw}\backslash LG_w/\mathrm{Iw} \xleftarrow{p_w} \frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})} \xrightarrow{q_w} \frac{LG}{\mathrm{Ad}_\theta(LG)}$.

4.4. **Beck-Chevalley condition for $\mathrm{Tr}(\Theta,\mathcal{H})$.** We first introduce some notations that will be used in the proofs in this subsection. For $n \geqslant 1$ and $1 \leqslant i < j \leqslant n$ we let:

$$W_n := X^n, \quad \text{and} \quad W_{n;i} := X^{i-1} \times Z \times X^{n-i},$$

and the morphisms:

$$a_{n;\underline{i}} : W_{n;i} \longrightarrow W_{n+1} \text{ (resp. } b_{n;\underline{i}} : W_{n;i} \longrightarrow W_n)$$

[8] This should be denoted by $(i_{\overline{\mathcal{O}'}})^* \circ F(\mathscr{F})$ we drop F from the notation since it is fully faithful.

denote the morphism obtained by applying a (resp. b) to the Z factor and the identity in all other factors.

For $i \neq j$, consider $W_{n;i,j}$ and the morphisms $a_{n;\underline{i},j}, a_{n;i,\underline{j}}, b_{n;i,\underline{j}}$, and $b_{n;\underline{i},j}$ defined such that all of the following are pullback diagrams:

$$\begin{tikzcd} W_{n;i,j} \arrow[r, "b_{n;i,\underline{j}}"] \arrow[d, "b_{n;\underline{i},j}"'] & W_{n;i} \arrow[d, "b_{n;\underline{i}}"] \\ W_{n;j} \arrow[r, "b_{n;\underline{j}}"'] & W_n \end{tikzcd}, \quad \begin{tikzcd} W_{n;i,j} \arrow[r, "b_{n;i,\underline{j}}"] \arrow[d, "a_{n;\underline{i},j}"'] & W_{n;i} \arrow[d, "a_{n;\underline{i}}"] \\ W_{n+1,j+1} \arrow[r, "b_{n+1;\underline{j+1}}"'] & W_{n+1} \end{tikzcd}, \text{ and } \begin{tikzcd} W_{n;i,j} \arrow[r, "b_{n;\underline{i},j}"] \arrow[d, "a_{n;i,\underline{j}}"'] & W_{n;j} \arrow[d, "a_{n;\underline{j}}"] \\ W_{n+1,i} \arrow[r, "b_{n+1;\underline{i}}"'] & W_{n+1} \end{tikzcd}.$$

We will also need:

$$\theta_k : W_n \longrightarrow W_n \text{ and } \theta_k : W_{n;i} \longrightarrow W_{n;i},$$

which applies the automorphism θ to the k-th factor of X, where $k \neq i$. We abuse notation and do not distinguish between these morphisms to keep the notation light.

The next two Lemmas check the part of Beck–Chevalley conditions that only involve the original simplicial object given by the twisted cyclic bar construction.

Lemma 4.4. *For $n \geqslant 0$ and $0 \leqslant i \leqslant n$, the degeneracy morphism $c_{i,n+1} : \mathcal{H}_n \to \mathcal{H}_{n+1}$ admits a right adjoint $c^{\mathrm{R}}_{i,n+1}$ and the induced diagram*

$$\begin{tikzcd} \mathcal{H}_{n+1} \arrow[r, "d_{0,n}"] & \mathcal{H}_n \\ \mathcal{H}_{n+2} \arrow[r, "d_{0,n+1}"'] \arrow[u, "c^{\mathrm{R}}_{i+1,n+2}"] & \mathcal{H}_{n+1} \arrow[u, "c^{\mathrm{R}}_{i,n+1}"'] \end{tikzcd} \tag{4.6}$$

commute.

Proof. Let $e : \operatorname{Spec} \mathbf{k} \to LG$ be the identity morphism, it induces an ind-fp proper map $\imath : \frac{\mathrm{pt}}{\mathrm{Iw}} \hookrightarrow \mathrm{Iw}\backslash LG/\mathrm{Iw}$, where $\frac{\mathrm{pt}}{\mathrm{Iw}} \simeq \mathrm{Iw}\backslash LG_{\leqslant 0}/\mathrm{Iw}$ and $0 \in \breve{W}$ is the identity element. Thus, the pushforward $\imath_*$ is well-defined (see Lemma 1.17). The unity in $D(X)$ is given by $\mathscr{F}_0 := \imath_*(\omega_{\frac{\mathrm{pt}}{\mathrm{Iw}}})$. Thus the functor $c_{i,n+1} : D(W_{n+1}) \to D(W_{n+2})$ is given by:

$$(\mathscr{F}_1, \ldots, \mathscr{F}_{n+1}) \longmapsto (\mathscr{F}_1, \ldots, \mathscr{F}_{i-1}, \mathscr{F}_0, \mathscr{F}_i, \ldots, \mathscr{F}_{n+1}).$$

Notice that the map $\pi : \frac{\mathrm{pt}}{\mathrm{Iw}} \to \mathrm{pt}$ is weakly cohomologically pro-smooth, since as discussed in the beginning of §4.3 the cover $\mathrm{pt} \to \mathrm{pt}/\mathrm{Iw}$ is essentially pro-unipotent. Thus, by Proposition 1.21 one has an adjuction $(\pi^!, \pi_*^{\mathrm{ren}})$.

Thus, we have the adjunction:

$$\mathrm{Hom}_{X^2}((\mathrm{id}_X \times \imath)_*(\mathrm{id}_X \times \pi)^!(\mathscr{F}), \mathscr{G}_1 \boxtimes \mathscr{G}_2) \simeq \mathrm{Hom}_{X^2}(\mathscr{F}, (\mathrm{id}_X \times \pi)_*(\mathrm{id}_X \times \imath)^!(\mathscr{G}_1 \boxtimes \mathscr{G}_2)),$$

from which it is clear that the right adjoint $c^{\mathrm{R}}_{i,n+1}$ to $c_{i,n+1}$ is given by

$$\begin{aligned} c^{\mathrm{R}}_{i,n+1} &: D(W_{n+2}) \longrightarrow D(W_{n+1}) \\ &(\mathscr{F}_1, \ldots, \mathscr{F}_{n+2}) \longmapsto (\mathscr{F}_1, \ldots, \mathscr{F}_{i-1}, (\mathrm{id}_X \times \pi)_*(\mathrm{id}_X \times \imath)^!(\mathscr{F}_i \boxtimes \mathscr{F}_{i+1}), \mathscr{F}_{i+2}, \cdots, \mathscr{F}_{n+2}). \end{aligned}$$

Now we check that the diagram (4.6) commutes. Let $\imath_{n;\underline{i}} : W_n \to W_{n+1}$ denote the inclusion of the identity on the $(i+1)$-th factor on the target. Notice that $c^{\mathrm{R}}_{i,n+1} \circ d_{0,n+1}$

(resp. $d_{0,n} \circ c^{\mathrm{R}}_{i+1,n+2}$) is given by pull-push along the bottom then the right side (resp. the left then the upper side) of the following diagram:

$$
\begin{tikzcd}
W_{n+2} & W_{n+1;n+1} \arrow[l, "\theta_{n+1}\circ a_{n+1,\underline{n+1}}"'] \arrow[r, "b_{n+1;\underline{n+1}}"] & W_{n+1} \\
W_{n+2} \times \mathrm{pt}/\breve{I} \arrow[u, "\mathrm{id}_{W_{n+2}}\times\pi"] \arrow[d, "\imath_{n+2,\underline{i+2}}"'] & W_{n+1;n+1} \times \mathrm{pt}/\breve{I} \arrow[l, "\theta_{n+1}\circ a_{n+1;\underline{n+1}}"'] \arrow[r, "b_{n+1,\underline{n+1}}"] \arrow[u, "\mathrm{id}_{W_{n+1;n+1}}\times\pi"'] \arrow[d, "\imath_{n+1,\underline{i+1}}"] & W_{n+1} \times \mathrm{pt}/\breve{I} \arrow[u, "\mathrm{id}_{W_{n+1}}\times\pi"'] \arrow[d, "\imath_{n+1,\underline{i+1}}"] \\
W_{n+3} & W_{n+2;n+2} \arrow[l, "\theta_{n+2}\circ a_{n+2;\underline{n+2}}"'] \arrow[r, "b_{n+2;\underline{n+2}}"] & W_{n+2}
\end{tikzcd}
\quad . \tag{4.7}
$$

We obtain the chain of isomorphisms:

$$
\begin{array}{c}
(b_{n+1,\underline{n+1}})_* \circ a^!_{n+1,\underline{n+1}} \circ \theta^!_{n+1} \circ (\mathrm{id}_{W_{n+2}} \times \pi)^{\mathrm{ren}}_* \circ \imath^!_{n+2,\underline{i+2}} \\
\simeq \big\downarrow \\
(b_{n+1,\underline{n+1}})_* \circ (\mathrm{id}_{W_{n+1;n+1}} \times \pi)^{\mathrm{ren}}_* \circ a^!_{n+1,\underline{n+1}} \circ \theta^!_{n+1} \circ \imath^!_{n+2,\underline{i+2}} \\
\big\downarrow \simeq \\
(\mathrm{id}_{W_{n+1}} \times \pi)^{\mathrm{ren}}_* \circ (b_{n+1,\underline{n+1}})_* \circ \imath^!_{n+1,\underline{i+1}} \circ a^!_{n+2;\underline{n+2}} \circ \theta^!_{n+2} \\
\big\downarrow \simeq \\
(\mathrm{id}_{W_{n+1}} \times \pi)^{\mathrm{ren}}_* \circ \imath^!_{n,\underline{i+1}} \circ (b_{n+2,\underline{n+2}})_* \circ a^!_{n+2;\underline{n+2}} \circ \theta^!_{n+2},
\end{array}
$$

where the first isomorphism is base change for the upper-left square of (4.7) by Proposition 1.21 (i), the second isomorphism is Proposition 1.21 (iii), and the third isomorphism is base change for the bottom-right square of (4.7) given by Lemma 1.17 (i).

This finishes the proof. □

Lemma 4.5. *For $n \geqslant 0$ and $0 \leqslant i \leqslant n$, the face morphism $d_{i,n} : \mathcal{H}_{n+1} \to \mathcal{H}_n$ admits a right adjoint $d^{\mathrm{R}}_{i,n}$ and the induced diagram*

$$
\begin{tikzcd}
\mathcal{H}_{n+2} \arrow[r, "d_{0,n+1}"] & \mathcal{H}_{n+1} \\
\mathcal{H}_{n+1} \arrow[u, "d^{\mathrm{R}}_{i+1,n+1}"] \arrow[r, "d_{0,n}"'] & \mathcal{H}_n \arrow[u, "d^{\mathrm{R}}_{i,n}"']
\end{tikzcd}
\tag{4.8}
$$

commutes.

Proof. Consider the commutative diagrams:

$$
\begin{tikzcd}
W_{n+3} & W_{n+2;n+2} \arrow[l, "\theta_{n+2}\circ a_{n+2;\underline{n+2}}"'] \arrow[r, "b_{n+2;\underline{n+2}}"] & W_{n+2} \\
W_{n+2,i+1} \arrow[u, "(\theta_{n+2}\circ)a_{n+2,\underline{i+1}}"] \arrow[d, "b_{n+2;\underline{i}}"'] & W_{n+1;i,n} \arrow[l, "\theta_{n+1}\circ a_{n+1;i,\underline{n+1}}"'] \arrow[r, "b_{n+1;i,\underline{n+1}}"] \arrow[u, "a_{n+1;\underline{i},n+1}"'] \arrow[d, "b_{n+1;\underline{i},n}"] & W_{n+1,i} \arrow[u, "(\eta\circ)a_{n+1,\underline{i}}"'] \arrow[d, "b_{n+1;\underline{i}}"] \\
W_{n+2} & W_{n+1;n+1} \arrow[l, "\theta_{n+1}\circ a_{n+1;\underline{n+1}}"'] \arrow[r, "b_{n+1;\underline{n+1}}"] & W_{n+1}
\end{tikzcd}
\tag{4.9}
$$

where $\eta : W_{n+2} \to W_{n+2}$ is the cyclic permutation that sends the first factor into the second factor, and so on. Here we use the notation:

$$
(\theta_{n+2}\circ)a_{n+2,\underline{i+1}} = \begin{cases} a_{n+2,\underline{i+1}}, & \text{if } i < n+1, \\ \theta_{n+2} \circ a_{n+2;\underline{n+2}}, & \text{if } i = n+1; \end{cases}
$$

and

$$
(\eta\circ)a_{n+1,\underline{i}} = \begin{cases} a_{n+1,\underline{i}}, & \text{if } i < n+1, \\ \eta \circ a_{n+1;\underline{n+1}}, & \text{if } i = n+1. \end{cases}
$$

Notice that for any $m \geqslant 1$, one has:

$$
\begin{aligned}
d_{i,0} &= (b_{m+1,\underline{m+1}})_* \circ a^!_{m+1;\underline{m+1}} \circ \theta^!_{m+1} \\
d_{i,m} &= (b_{m+1,\underline{i}})_* \circ a^!_{m+1;\underline{i+1}} \text{ for } 0 < i < m+1 \\
d_{m+1,m} &= (b_{m+1,\underline{m+1}})_* \circ a^!_{m+1;\underline{m+1}} \circ \eta^!.
\end{aligned}
$$

Define:

$$
\begin{aligned}
d^{\mathrm{R}}_{i,0} &:= (\theta_{m+1})^{\mathrm{ren}}_* \circ (a_{m+1;\underline{i+1}})^{\mathrm{ren}}_* \circ b^!_{m+1,\underline{i}} \\
d^{\mathrm{R}}_{i,m} &:= (a_{m+1;\underline{i+1}})^{\mathrm{ren}}_* \circ b^!_{m+1,\underline{i}} \text{ for } 0 < i < m+1 \\
d^{\mathrm{R}}_{m+1,m} &:= \eta^{\mathrm{ren}}_* \circ (a_{m+1;\underline{m+1}})^{\mathrm{ren}}_* \circ b^!_{m+1,\underline{m+1}}.
\end{aligned}
$$

For all $m \geqslant 0$ and $0 \leqslant i \leqslant m+1$, the fibers of the morphism $a_{m+1;\underline{i}} : W_{m+1;i} \to W_{m+2}$ are isomorphic to the essentially pro-unipotent scheme Iw. Hence $a_{m+1;\underline{i}}$ is essentially pro-unipotent, thus $(a_{m+1;\underline{i}})^{\mathrm{ren}}_*$ is well-defined and we have an adjunction:

$$
a^!_{m+1;\underline{i}} : D(W_{m+1}) \rightleftarrows D(W_{m+2}) : (a_{m+1;\underline{i}})^{\mathrm{ren}}_* .
$$

The morphisms $b_{m+1;\underline{i}} : W_{m+1;\underline{i}} \to W_{m+1}$ are ind-fp-proper, since their fibers are isomorphic to Fl, so we obtain the adjunction:

$$
(b_{m+1;\underline{i}})_* : D(W_{m+1;\underline{i}}) \rightleftarrows D(W_{m+1}) : b^!_{m+1;\underline{i}} .
$$

Since η and θ_m are isomorphisms. We obtain that $d^{\mathrm{R}}_{i,m}$ is right adjoint to $d_{i,m}$.

Now we check that the diagram (4.8) commutes. Notice that the top left and bottom right squares in (4.9) are Cartesian. The natural base change map:

$$\begin{tikzcd}
(b_{n+2;\underline{n+2}})_* \circ a^!_{n+2;\underline{n+2}} \circ \theta^!_{n+2} \circ (a_{n+2;\underline{i+1}})^{\mathrm{ren}}_* \circ b^!_{n+2;\underline{i}} \arrow[d, "\simeq"] \\
(b_{n+2;\underline{n+2}})_* \circ (a_{n+1;\underline{i},n+1})^{\mathrm{ren}}_* \circ a^!_{n+1;i,\underline{n+1}} \circ \theta^!_{n+1} \circ b^!_{n+2;\underline{i}} \arrow[d, "\simeq"] \\
(a_{n+1;\underline{i}})^{\mathrm{ren}}_* \circ (b_{n+1;i,\underline{n+1}})_* \circ b^!_{n+1;\underline{i},n} \circ a^!_{n+1;\underline{n+1}} \circ \theta^!_{n+1} \arrow[d, "\simeq"] \\
(a_{n+1;\underline{i}})^{\mathrm{ren}}_* \circ b^!_{n+1;\underline{i}} \circ (b_{n+1;\underline{n+1}})_* \circ a^!_{n+1;\underline{n+1}} \circ \theta^!_{n+1}
\end{tikzcd}$$

is an isomorphism. Indeed, the first arrow is an isomorphism by Proposition 1.21 (i) and since since $b_{n+1;\underline{i}}$ is ind-fp-proper the the middle arrow is an isomorphism by Proposition 1.21 (iii) and the last arrow is an isomorphism by Lemma 1.17 (i). We have similar isomorphisms when we include θ_{n+2} or η in the edge cases. $\square$

Now we need to define an augmented simplicial object $\mathcal{H}^+_\bullet : \Delta^{\mathrm{op}}_+ \to \mathrm{Lincat}_E$, whose restriction to Δ^{op} is the twisted cyclic Bar construction on $\mathcal{H}$ (see (4.2)). In our situation, a natural candidate for the augmentation extension is (see (4.4) for the definition of CH):

$$CH : \mathcal{H}_0 \longrightarrow \mathcal{H}^+_{-1} := D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right).$$

Lemma 4.6. *The functor $\mathcal{H}^+_\bullet : \Delta^{\mathrm{op}}_+ \to \mathrm{Lincat}_E$ extends the twisted cyclic bar construction on $\mathcal{H}$ to an augmented simplicial object. Moreover, the diagram:*

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[r, "d_{0,0}"] \arrow[d, "d_{1,0}"'] & \mathcal{H}_0 \arrow[d, "d_{0,-1}"] \\
\mathcal{H}_0 \arrow[r, "d_{0,-1}"'] & \mathcal{H}^+_{-1}
\end{tikzcd} \tag{4.10}$$

is vertically right adjointable.

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
(\mathrm{Iw}\backslash LG/\mathrm{Iw})^2 & \mathrm{Iw}\backslash LG \times^{\mathrm{Iw}} LG/\mathrm{Iw} \arrow[l, "\theta_1 \circ a"'] \arrow[r, "b"] & \mathrm{Iw}\backslash LG/\mathrm{Iw} \\
\mathrm{Iw}\backslash LG \times^{\mathrm{Iw}} LG/\mathrm{Iw} \arrow[u, "\sigma\circ a"] \arrow[d, "b"'] & \frac{LG\times^{\mathrm{Iw}}LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[l, "\overrightarrow{\mu}"'] \arrow[r, "\overleftarrow{h}"] \arrow[u, "\overleftarrow{\mu}"'] \arrow[d, "\overrightarrow{h}"] & \frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[u, "p"'] \arrow[d, "q"] \\
\mathrm{Iw}\backslash LG/\mathrm{Iw} & \frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[l, "p"'] \arrow[r, "q"] & \frac{LG}{\mathrm{Ad}_\theta(LG)}
\end{tikzcd} \tag{4.11}$$

where σ is the swap morphism and θ_1 applies the automorphism θ on the first factor. Let $[(g,g')] \in \frac{LG\times^{\mathrm{Iw}}LG}{\mathrm{Ad}_\theta(\mathrm{Iw})}$, where $(g,g') \in LG \times LG$. The morphisms $\overrightarrow{h}$, $\overleftarrow{h}$ are define in (2.4), where $\overrightarrow{\mu}$ and $\overleftarrow{\mu}$ are given by:

$$\overrightarrow{\mu}([g,g']) = ([g',\theta(g)]) \quad \text{and} \quad \overleftarrow{\mu}([g,g']) = ([g,g']),$$

Thus, one has:

$$d_{0,-1}\circ d_{0,0} = q_*\circ p^!\circ b_*\circ a^!\circ \theta_1^! \text{ and } d_{0,-1}\circ d_{1,0} = q_*\circ p^!\circ b_*\circ \sigma^!\circ a^!.$$

One can directly check that all the squares in (4.11) are Cartesian. Since b is ind-fp-proper, by Lemma 1.17 (i) the commutativity of the diagram (4.10) follows from the base change isomorphisms:

$$q_*\circ(\overleftarrow{h})_*\circ(\overleftarrow{\mu})^!\circ a^!\circ\theta_1^! \xrightarrow{\sim} q_*\circ p^!\circ b_*\circ\theta_1^!\circ a^!;$$
$$q_*\circ(\overrightarrow{h})_*\circ(\overrightarrow{\mu})^!\circ \sigma^!\circ a^! \xrightarrow{\sim} q_*\circ p^!\circ b_*\circ\sigma^!\circ a^!.$$

Moreover, since a and p are weakly cohomologicall pro-smooth, one has functos:

$$\sigma_*^{\mathrm{ren}}\circ a_*^{\mathrm{ren}}\circ b^! : D(X)\longrightarrow D(X^2), \text{ and } p_*^{\mathrm{ren}}\circ q^! : D(Y)\longrightarrow D(X)$$

which are right adjoints to $b_*\circ\sigma^!\circ a^!$ and $q_*\circ p^!$, respectively. Thus, to check that the diagram (4.10) is vertically right adjointable we need to check that the canonical morphism:

$$\begin{aligned} b_*\circ a^!\circ\sigma^!\circ(\theta_1)_*^{\mathrm{ren}}\circ a_*^{\mathrm{ren}}\circ b^! &\longrightarrow b_*\circ(\overrightarrow{\mu})_*^{\mathrm{ren}}\circ(\overleftarrow{\mu})^!\circ b^! \\ &\simeq p_*^{\mathrm{ren}}\circ(\overrightarrow{h})_*\circ(\overleftarrow{h})^!\circ p^! \\ &\longrightarrow p_*^{\mathrm{ren}}\circ q^!\circ q_*\circ p^! \end{aligned}$$

is an isomorphism. The first arrow is an isomorphism by Proposition 1.21 (i), the middle one follows from Proposition 1.21 (iii) and the last by Lemma 1.17 (i). □

4.5. **Stratification of** $\mathrm{Tr}(\Theta,\mathcal{H})$**.** In this subsection, we get a stratification of the subcategory $\mathrm{Tr}(\Theta,\mathcal{H})$ from the stratification of $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$.

We start with the following observation:

Lemma 4.7. *The functor (4.5) makes* $\mathrm{Tr}(\Theta,\mathcal{H})$ *into an open subcategory of* $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$.

Proof. We need to check that we have the following recollement diagram:

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "\mathrm{tr}^{\mathrm{enh.}}"] \arrow[r, bend right, dashed, "\mathrm{tr}^{\mathrm{enh.,RR}}"'] & \mathrm{Tr}(\Theta,\mathcal{H}) \arrow[l, dashed, "\mathrm{tr}^{\mathrm{enh.,R}}"'] \arrow[r, bend left, "F"] \arrow[r, bend right, dashed, "F^{\mathrm{RR}}"'] & D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right) \arrow[l, dashed, "F^{\mathrm{R}}"'] \end{tikzcd}.$$

First, the functors $\mathrm{tr}^{\mathrm{enh.,R}}$ and F^{R} exist as a consequence of Proposition 4.1.

Notice that $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$ and $\mathcal{H}$ are compactly generated (see Remark 1.16 2). Moreover, by construction the functor (4.3) enconding the algebra structure of $\mathcal{H}$ and similarly the functor enconding the (twisted) action of $\mathcal{H}$ on itself preserve compact objects. Hence [10, Chapter 1, Corollary 8.7.4] implies that $\mathrm{Tr}(\Theta,\mathcal{H})$ is also compactly generated. Since the composite $\mathrm{tr}^{\mathrm{R}} := \mathrm{tr}^{\mathrm{enh.,R}}\circ F^{\mathrm{R}} = p_*^{\mathrm{ren}}\circ q^!$ is continuous by construction (see Proposition 1.21), compact generation implies that the functors $\mathrm{tr}^{\mathrm{enh.,R}}$ and F^{R} admit further right adjoints. □

To check that the stratification of $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$ restricts to a stratification of $\mathrm{Tr}(\Theta, \mathcal{H})$ it is convenient to use a criterion established in [3].

Proposition 4.8. *[3, Proposition 3.4.7] Let $\mathcal{U}_\bullet : \mathsf{P} \to \mathcal{X}$ be an open stratification of $\mathcal{X}$ and let $\imath : \mathcal{Y} \hookrightarrow \mathcal{X}$ be an open subcategory, such that for every $p \in \mathsf{P}$ we have factorizations:*

$$(i)\quad \begin{tikzcd} \mathcal{U}_p \cap \mathcal{Y} \arrow[r, hook] & \mathcal{Y} \\ \mathcal{U}_p \arrow[r, hook, "\jmath_p"'] \arrow[u, dashed] & \mathcal{X} \arrow[u, "\imath^{\mathrm{R}}"'] \end{tikzcd}; \qquad (ii)\quad \begin{tikzcd} \mathcal{U}_p \cap \mathcal{Y} \arrow[r, hook] & \mathcal{U}_p \\ \mathcal{Y} \arrow[r, hook, "\imath"'] \arrow[u, dashed] & \mathcal{X} \arrow[u, "\jmath_p^{\mathrm{R}}"'] \end{tikzcd}.$$

Then the assignment:

$$\mathsf{P} \longrightarrow \mathrm{Op}_{\mathcal{Y}}, \qquad P \longmapsto \mathcal{U}_p \cap \mathcal{Y}$$

determines an open P-stratification of $\mathcal{Y}$. Moreover, the pth stratum is given by $\mathcal{X}_p \cap \mathcal{Y}$, where $\mathcal{X}_p$ is the pth stratum of the stratification of $\mathcal{X}$.

To apply this result to the decomposition of $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$ we need to rephrase the closed stratification of Theorem 3.5 as an open stratification. We will follow the construction from §A.3. For every $\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}$, let

$$Z_{\mathcal{O}} := \bigcup_{\mathcal{O}' \not\geq \mathcal{O}} LG_{\overline{\mathcal{O}'}} = \bigsqcup_{\mathcal{O}' \not\geq \mathcal{O}} LG_{\mathcal{O}'}, \quad \text{and} \quad LG_{\gneq \mathcal{O}} := \bigcup_{\mathcal{O}' \gneq \mathcal{O}} LG_{\mathcal{O}'}. \tag{4.12}$$

We denote by

$$p_{\mathcal{O}} : \frac{Z_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)} \hookrightarrow \frac{LG}{\mathrm{Ad}_\theta(LG)} \quad \text{and} \quad i_{\gneq \mathcal{O}} : \frac{LG_{\gneq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)} \hookrightarrow \frac{LG}{\mathrm{Ad}_\theta(LG)}$$

the induced fp-closed embedding and qc qs open complement. Notice that by Lemma 3.2, we have

$$\bigcup_{\mathcal{O}' \not\geq \mathcal{O}} D\left(\frac{LG_{\overline{\mathcal{O}'}}}{\mathrm{Ad}_\theta(LG)}\right) \xrightarrow{\sim} D\left(\frac{Z_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right).$$

Thus, by applying the open-closed correspondence of §A.3 to the closed stratification (3.11) we obtain the category $\mathsf{R}\left(D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)\right)_{\mathcal{O}}$, which can be concretely described as:

$$\left\{\mathscr{F} \in D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right) \mid \mathrm{Map}_{D(\frac{LG}{\mathrm{Ad}_\theta(LG)})}(\mathscr{F}, (p_{\mathcal{O}})_*\mathscr{G}) \text{ for all } \mathscr{G} \in D\left(\frac{Z_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right)\right\}.$$

By the adjunction $((p_{\mathcal{O}})^*, (p_{\mathcal{O}})_*)$ and (1.31) we have that $\mathscr{F} \simeq (i_{\gneq \mathcal{O}})_!(\mathscr{F}_{\gneq \mathcal{O}})$ for some $\mathscr{F}_{\gneq \mathcal{O}} \in D\left(\frac{LG_{\gneq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right)$.

This implies that for every $\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}$, we have the following recollement diagram:

$$\begin{tikzcd} D\left(\frac{Z_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[r, hook, "(p_{\mathcal{O}})_*" description] & D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right) \arrow[l, bend right, "p_{\mathcal{O}}^*"'] \arrow[l, bend left, "p_{\mathcal{O}}^!"] \arrow[r, "(i_{\gneq \mathcal{O}})^!" description] & D\left(\frac{LG_{\gneq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[l, bend right, hook, "(i_{\gneq \mathcal{O}})_!"'] \arrow[l, bend left, hook, "(i_{\gneq \mathcal{O}})_*"] \end{tikzcd}.$$

By Proposition A.6 one obtains an open $(\breve{W} /\!\!/_\theta \breve{W})^{\mathrm{op}}$-stratification of $D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)$:

$$(\breve{W} /\!\!/_\theta \breve{W})^{\mathrm{op}} \longrightarrow \mathrm{Op}_{D(\frac{\breve{G}}{\mathrm{Ad}_\theta(\breve{G})})}, \qquad \mathcal{O} \longmapsto D\left(\frac{LG_{\succcurlyeq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right).$$

Now for every $\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}$ we let:

$$\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}} := D\left(\frac{LG_{\succcurlyeq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \cap \mathrm{Tr}(\Theta, \mathcal{H}).$$

Concretely, we have:

$$\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}} \simeq \left\{ \mathscr{F} \in \mathrm{Tr}(\mathcal{H}, \Theta) \mid F(\mathscr{F}) \simeq (i_{\succcurlyeq \mathcal{O}})_!(\mathscr{G}) \text{ for some } \mathscr{G} \in D\left(\frac{LG_{\succcurlyeq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right)\right\} \tag{4.13}$$

and Lemma 1.18 implies that we also have:

$$\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}} \simeq \{\mathscr{F} \in \mathrm{Tr}(\mathcal{H}, \Theta) \mid (p_{\mathcal{O}})^* \circ F(\mathscr{F}) = 0\}. \tag{4.14}$$

To obtain an open $(\breve{W} /\!\!/_\theta \breve{W})^{\mathrm{op}}$-stratification of $\mathrm{Tr}(\Theta, \mathcal{H})$ we need to check conditions (i) and (ii) from Proposition 4.8. Condition (i) uses the details of the definiton of $LG_{\overline{\mathcal{O}}}$, whereas (ii) requires some nontrivial results on what the Harish-Chandra morphism does to sheaves coming from a given strata in the affine Hecke category (see Lemma 4.10 below).

Lemma 4.9. *For every $\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}$ we have the following factorization:*

$$\begin{tikzcd}
\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}} \arrow[r, hook] & \mathrm{Tr}(\Theta, \mathcal{H}) \\
D\left(\frac{LG_{\succcurlyeq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[u, dashed] \arrow[r, hook, "\pi^{\mathrm{L}}_{\mathcal{O}}"'] & D\left(\frac{\breve{G}}{\mathrm{Ad}_\theta(\breve{G})}\right) \arrow[u, "F^{\mathrm{R}}"']
\end{tikzcd}.$$

Proof. The functor F^{R} is hard to describe directly, so we use the following trick. Let $\mathcal{H}_{\mathcal{O}} := \{\mathscr{F} \in \mathcal{H} \mid (p_{\mathcal{O}})^* \circ CH(\mathscr{F}) = 0\}$. Notice that the restriction of tr to $\mathcal{H}_{\mathcal{O}}$ factors through $\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}}$ and $D\left(\frac{LG_{\succcurlyeq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right)$. So we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{O}} \arrow[r, hook, "\pi^{\mathrm{L}}_{\mathcal{H},\mathcal{O}}"] \arrow[d, "\mathrm{tr}_{\mathcal{O}}"'] & \mathcal{H} \arrow[d, "\mathrm{tr}"] \\
\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}} \arrow[r, hook, "\pi^{\mathrm{L}}_{\mathrm{Tr},\mathcal{O}}"] \arrow[d, hook, "F_{\mathcal{O}}"'] & \mathrm{Tr}(\Theta, \mathcal{H}) \arrow[d, hook, "F"] \\
D\left(\frac{LG_{\succcurlyeq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[r, hook, "(i_{\succcurlyeq \mathcal{O}})_!"] & D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right)
\end{tikzcd}, \tag{4.15}$$

where the top two horizontal arrows are simply the canonical inclusions. By passing to right adjoints for the vertical maps we obtain that the following lax-commutative diagram:

$$\begin{tikzcd} \mathcal{H}_{\mathcal{O}} \arrow[r, hook, "\pi^{\mathrm{L}}_{\mathcal{H},\mathcal{O}}"] \arrow[dr, phantom, "\Rrightarrow"] & \mathcal{H} \\ \mathrm{Tr}(\Theta,\mathcal{H})_{\succeq\mathcal{O}} \arrow[u, "(\mathrm{tr}_{\mathcal{O}})^{\mathrm{R}}"] \arrow[r, hook, "\pi^{\mathrm{L}}_{\mathrm{Tr},\mathcal{O}}"] \arrow[dr, phantom, "\Rrightarrow"] & \mathrm{Tr}(\Theta,\mathcal{H}) \arrow[u, "\mathrm{tr}^{\mathrm{R}}"'] \\ D\left(\frac{LG_{\succcurlyeq\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \arrow[u, "F^{\mathrm{R}}_{\mathcal{O}}"] \arrow[r, hook, "(i_{\succcurlyeq\mathcal{O}})_!"'] & D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right) \arrow[u, "F^{\mathrm{R}}"'] \end{tikzcd} \tag{4.16}$$

Since $\mathcal{H}_{\mathcal{O}}$ generates $\mathrm{Tr}(\Theta,\mathcal{H})_{\succeq\mathcal{O}}$, the diagram (4.15) directly implies that the upper square of (4.16) commutes.

Now we claim that the outer diagram of (4.16) also commutes. Concretely, we need to check that $CH^R \circ (i_{\succcurlyeq\mathcal{O}})_! : D\left(\frac{LG_{\succcurlyeq\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \to \mathcal{H}$ factors through $\mathcal{H}_{\mathcal{O}}$, i.e. that

$$(p_{\mathcal{O}})^* \circ CH \circ CH^R \circ (i_{\succcurlyeq\mathcal{O}})_! = 0.$$

By (4.12) this is equivalent to checking that

$$(i_{\overline{\mathcal{O}'}})^* \circ q_* \circ p^! \circ p_*^{\mathrm{ren}} \circ q^! \circ (i_{\succcurlyeq\mathcal{O}})_! = 0 \text{ for all } \mathcal{O}' \not\succcurlyeq \mathcal{O}.$$

Let $a_{\overline{\mathcal{O}'}} : LG_{\overline{\mathcal{O}'}} \to \frac{LG_{\overline{\mathcal{O}'}}}{\mathrm{Ad}_\theta(\mathrm{Iw})} \to \frac{LG_{\overline{\mathcal{O}'}}}{\mathrm{Ad}_\theta(LG)}$ be the natural quotient map, by descent (1.24) and Remark 1.16 (1), it is enough to check that

$$(a_{\overline{\mathcal{O}'}})^! \circ (i_{\overline{\mathcal{O}'}})^* \circ q_* \circ p^! \circ p_*^{\mathrm{ren}} \circ q^! \circ (i_{\succcurlyeq\mathcal{O}})_! = 0.$$

Let $w' \in \breve{W}$ be a minimal length representative in the orbit $[w']_\theta = \mathcal{O}'$ and recall the presentation (2.8) and let $i^{\overline{\mathcal{O}'}}_{u,\leqslant[w']_\theta} : LG_{u,\leqslant[w']_\theta} \hookrightarrow LG_{\overline{\mathcal{O}'}}$. To check that a sheaf $\mathscr{F} \in D(LG_{\overline{\mathcal{O}'}})$ vanishes, it is enough to check that $(i^{\overline{\mathcal{O}'}}_{u,\leqslant[w']_\theta})^! \mathscr{F}$ vanishes for every $u \in \breve{W}$. Indeed, it is enough to check that the $*$-restriction of $\mathscr{F}$ to each fp-closed $LG_{\leqslant u,\leqslant[w']_\theta} \hookrightarrow LG_{\overline{\mathcal{O}'}}$ vanishes, and since $LG_{\leqslant u,\leqslant[w']_\theta} = \sqcup_{v'\leqslant u,\leqslant[w']_\theta}$, by induction it is enough to check that the $*$-restrictions to each locally closed $LG_{u,\leqslant[w']_\theta}$ vanishes. Namely, that for every $u \in \breve{W}$ we need to prove that:

$$(f_{u,w'})^* \circ q^! \circ q_* \circ p^! \circ p_*^{\mathrm{ren}} \circ q^! \circ (i_{\succcurlyeq\mathcal{O}})_! \simeq 0 \text{ for all } \mathcal{O}' \not\succcurlyeq \mathcal{O}. \tag{4.17}$$

Consider the diagram:

$$\begin{tikzcd} \mathrm{Iw}\backslash(LG_u \times^{\mathrm{Iw}} LG_{\leqslant w'})/\mathrm{Iw} \arrow[r, "f^{\mathcal{H}}_{u,w'}"] & \mathrm{Iw}\backslash(LG \times^{\mathrm{Iw}} LG)/\mathrm{Iw} \arrow[r, "b"] & \mathrm{Iw}\backslash LG/\mathrm{Iw} \\ LG_u \times^{\mathrm{Iw}} FL_{\leqslant w'} \arrow[u, "\mu_{u,w'}"] \arrow[r, "f^{\mathrm{Iw}}_{u,w'}"] \arrow[d, "m_{u,w'}"] & \frac{LG\times^{\mathrm{Iw}}LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[u, "\mu"] \arrow[r, "\overrightarrow{h}"] \arrow[d, "\overleftarrow{h}"] & \frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[u, "p"] \arrow[d, "q"] \\ LG_{u,\leqslant[w']_\theta} \arrow[r, "f_{u,w'}"] & \frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[r, "q"] & \frac{LG}{\mathrm{Ad}_\theta(LG)} \end{tikzcd},$$

where all the squares are Cartesian. We notice that we have:

$$
\begin{aligned}
(f_{u,w'})^* \circ q^! \circ q_* \circ p^! \circ p_*^{\mathrm{ren}} \circ q^! &\simeq (f_{u,w'})^* \circ \overleftarrow{h}_* \circ \overrightarrow{h}^! \circ p^! \circ p_*^{\mathrm{ren}} \circ q^! \\
&\simeq (m_{u,w'})_* \circ (f_{u,w'}^{\mathrm{Iw}})^* \circ \overrightarrow{h}^! \circ p^! \circ p_*^{\mathrm{ren}} \circ q^! \\
&\simeq (m_{u,w'})_* \circ (\mu_{u,w'})^! \circ (f_{u,w'}^{\mathcal{H}})^* \circ b^! \circ p_*^{\mathrm{ren}} \circ q^! \\
&\simeq (m_{u,w'})_* \circ (\mu_{u,w'})^! \circ (f_{u,w'}^{\mathcal{H}})^* \circ \mu_*^{\mathrm{ren}} \circ \overrightarrow{h}^! \circ q^! \\
&\simeq (m_{u,w'})_* \circ (\mu_{u,w'})^! \circ (\mu_{u,w'})_*^{\mathrm{ren}} \circ (f_{u,w'}^{\mathrm{Iw}})^* \circ \overrightarrow{h}^! \circ q^! \\
&\simeq (m_{u,w'})_* \circ (\mu_{u,w'})^! \circ (\mu_{u,w'})_*^{\mathrm{ren}} \circ (m_{u,w'})^! \circ (f_{u,w'})^* \circ q^!,
\end{aligned}
$$

where the base change statement for $*$-pullback hold because the corresponding morphisms are open.

From the following commutative diagram:

$$
\begin{tikzcd}
LG_{u,\leqslant[w']_\theta} \arrow[d, "f_{u,w'}^{\overline{\mathcal{O}'}}"] \arrow[dr, "f_{u,w'}"] & \\
\frac{LG_{\overline{\mathcal{O}'}}}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[r, "i_{\overline{\mathcal{O}'}}^{\mathrm{Iw}}"] \arrow[d, "q_{\overline{\mathcal{O}'}}"] & \frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[d, "q"] \\
\frac{LG_{\overline{\mathcal{O}'}}}{\mathrm{Ad}_\theta(LG)} \arrow[r, "i_{\overline{\mathcal{O}'}}"] & \frac{LG}{\mathrm{Ad}_\theta(LG)}
\end{tikzcd},
$$

we obtain that:

$$
\begin{aligned}
(f_{u,w'})^* \circ q^! \circ (i_{\succcurlyeq\mathcal{O}})_! &\simeq (f_{u,w'}^{\overline{\mathcal{O}'}})^* \circ (i_{\overline{\mathcal{O}'}}^{\mathrm{Iw}})^* \circ q^! \circ (i_{\succcurlyeq\mathcal{O}})_! \\
&\simeq (f_{u,w'}^{\overline{\mathcal{O}'}})^* \circ (q_{\overline{\mathcal{O}'}})^! \circ i_{\overline{\mathcal{O}'}}^* \circ (i_{\succcurlyeq\mathcal{O}})_!
\end{aligned}
$$

which vanishes for $\mathcal{O}' \not\succcurlyeq \mathcal{O}$. This gives (4.17) and finishes the proof. $\square$

We separate the crucial calculation necessary to check condition (ii) of Proposition 4.8 in the following:

Lemma 4.10. *Let $w \in \breve{W}$ and $s \in \breve{\mathbb{S}}$. Consider the diagram:*

$$
\mathrm{Iw}\backslash LG_w/\mathrm{Iw} \xleftarrow{p_w} \frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})} \xrightarrow{q_w} \frac{LG}{\mathrm{Ad}_\theta(LG)}. \tag{4.18}
$$

Set $w' = sw\theta(s)$ and $\mathcal{F}_w := q_{w,*} \circ p_w^!(\omega_{\mathrm{Iw}\backslash LG_w/\mathrm{Iw}})$.

(1) If $\ell(w') = \ell(w)$, then $\mathcal{F}_w = \mathcal{F}_{w'}$.

(2) If $\ell(w') < \ell(w)$, then one has a cofiber-fiber sequence:

$$
\mathcal{F}_{sw}[2] \oplus \mathcal{F}_{sw}[1] \longrightarrow \mathcal{F}_w \longrightarrow \mathcal{F}_{w'}[2].
$$

Proof. Without loss of generality, we may assume that $sw < w$. Consider the following diagram:

$$
\begin{tikzcd}
\mathrm{Iw}\backslash LG_w/\mathrm{Iw} & \frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[l, "p_w"'] \arrow[dr, "q_w"] & \\
\mathrm{Iw}\backslash LG_s \times^{\mathrm{Iw}} LG_{sw}/\mathrm{Iw} \arrow[u, "\cong"', "m'"] \arrow[d, "\cong", "f'"'] & \frac{LG_s \times^{\mathrm{Iw}} LG_{sw}}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[l, "p'"'] \arrow[u, "\cong"', "m"] \arrow[d, "\cong", "f"'] \arrow[r, "q_w \circ m"] & \frac{LG}{\mathrm{Ad}_\theta(LG)}. \\
\mathrm{Iw}\backslash LG_{sw} \times^{\mathrm{Iw}} LG_{\theta(s)}/\mathrm{Iw} & \frac{LG_{sw} \times^{\mathrm{Iw}} LG_{\theta(s)}}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[l, "p''"'] \arrow[ur, "q_w \circ m"'] &
\end{tikzcd}
$$

Here $A \times^{\mathrm{Iw}} B$ is the quotient of $A \times B$ by the Iw-action defined by $i(a,b) = (ai^{-1}, ib)$, p' and p'' are the projection maps from $\frac{-}{\mathrm{Ad}_\theta(\mathrm{Iw})}$ to $\mathrm{Iw}\backslash - /\mathrm{Iw}$ associated to $\mathrm{Ad}_\theta(\mathrm{Iw}) \subset \mathrm{Iw} \times \mathrm{Iw}$, m and m' are the multiplication maps, and f, f' are induced from the swap map

$$LG_s \times LG_{sw} \longrightarrow LG_{sw} \times LG_s, \qquad (g_1, g_2) \longmapsto (g_2, \theta(g_1)).$$

Thus we have

$$
\begin{aligned}
\mathcal{F}_w &= (q_w)_* \circ p_w^!(\omega_{\mathrm{Iw}\backslash LG_w/\mathrm{Iw}}) = (q_w \circ m)_*(p')^!(\omega_{\mathrm{Iw}\backslash LG_s \times^{\mathrm{Iw}} LG_{sw}/\mathrm{Iw}}) \\
&= (q_w \circ m)_*(p'')^!(\omega_{\mathrm{Iw}\backslash LG_{sw} \times^{\mathrm{Iw}} LG_{\theta(s)}/\mathrm{Iw}}).
\end{aligned}
$$

If $\ell(w') = \ell(w)$, then the multiplication map gives an isomorphism $LG_{sw} \times^{\mathrm{Iw}} LG_{\theta(s)} \cong LG_{w'}$. Thus we have

$$\mathcal{F}_w = (q_w \circ m)_*(p'')^!(\omega_{\mathrm{Iw}\backslash LG_{sw} \times^{\mathrm{Iw}} LG_{\theta(s)}/\mathrm{Iw}}) = (q_{w'})_* \circ p_{w'}^!(\omega_{\mathrm{Iw}\backslash LG_{w'}/\mathrm{Iw}}) = \mathcal{F}_{w'}.$$

Now we consider the case where $\ell(w') < \ell(w)$. Consider the diagram:

$$
\begin{tikzcd}
\frac{X_1}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[r, hook, "\imath"] \arrow[d, "p_1"'] & \frac{LG_{sw} \times^{\mathrm{Iw}} LG_{\theta(s)}}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[d] & \frac{X_2}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[l, hook', "\jmath"'] \arrow[d, "p_2"] \\
\frac{LG_{sw}}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[r, hook, "\imath'"'] & \frac{LG_{sw,\theta(s)}}{\mathrm{Ad}_\theta(\mathrm{Iw})} & \frac{LG_{w'}}{\mathrm{Ad}_\theta(\mathrm{Iw})}, \arrow[l, hook', "\jmath'"]
\end{tikzcd}
$$

where:

$$LG_{sw,\theta(s)} = \begin{cases} LG_{w'} & \text{if } w' > sw; \\ LG_{w'} \sqcup LG_{sw} & \text{if } w' < sw, \end{cases}$$

$\jmath$ is an open embedding, and $\imath$ is the closed embedding from its complement. The projection $p_1 : \frac{X_1}{\mathrm{Ad}_\theta(\mathrm{Iw})} \to \frac{LG_{sw}}{\mathrm{Ad}_\theta(\mathrm{Iw})}$ is an $(\mathbf{A}^1\backslash\{0\})$-fibration and $p_2 : \frac{X_2}{\mathrm{Ad}_\theta(\mathrm{Iw})} \to \frac{LG_{w'}}{\mathrm{Ad}_\theta(\mathrm{Iw})}$ is an $\mathbf{A}^1$-fibration. Thus, we have:

$$p_{1,*}\omega_{\frac{X_1}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \simeq \omega_{\frac{LG_{sw}}{\mathrm{Ad}_\theta(\mathrm{Iw})}}[1] \oplus \omega_{\frac{LG_{sw}}{\mathrm{Ad}_\theta(\mathrm{Iw})}}[2]$$

and

$$p_{2,*}\omega_{\frac{X_2}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \simeq \omega_{\frac{LG_{w'}}{\mathrm{Ad}_\theta(\mathrm{Iw})}}[2].$$

By applying the localization sequence to the closed embedding of $\frac{X_1}{\mathrm{Ad}_\theta(\mathrm{Iw})}$ in $\frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})}$ gives:

$$\imath_*\imath^!\omega_{\frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \longrightarrow \omega_{\frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \longrightarrow \jmath_* \circ \jmath^!\omega_{\omega_{\frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})}}}.$$

We pushforward via p_w to obtain:

$$\begin{tikzcd}
p_{w,*}\imath_*\imath^!\omega_{\frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \arrow[r] \arrow[d, "\simeq"] & p_{w,*}\omega_{\frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \arrow[r] \arrow[d, "\simeq"] & p_{w,*}\jmath_* \circ \jmath^!\omega_{\frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \arrow[d, "\simeq"] \\
p_{\dot{s}\dot{w},*} \circ p_{1,*}\omega_{\frac{X_1}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \arrow[r] \arrow[d, "\simeq"] & \mathcal{F}_w \arrow[r] \arrow[d, "\simeq"] & p_{\dot{w}',*} \circ p_{2,*}\omega_{\frac{X_2}{\mathrm{Ad}_\theta(\mathrm{Iw})}} \arrow[d, "\simeq"] \\
\mathcal{F}_{\dot{s}\dot{w}}[2] \oplus \mathcal{F}_{\dot{s}\dot{w}}[1] \arrow[r] & \mathcal{F}_w \arrow[r] & \mathcal{F}_{\dot{w}'}[2]
\end{tikzcd}$$

This finishes the proof. □

Lemma 4.11. *For every $\mathcal{O} \in \breve{W} /\!\!/_\theta \breve{W}$ we have the following factorization:*

$$\begin{tikzcd}
\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}} \arrow[r, hook] & D\left(\frac{LG_{\succcurlyeq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \\
\mathrm{Tr}(\Theta, \mathcal{H}) \arrow[r, hook, "F"'] \arrow[u, dashed] & D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right) \arrow[u, "(i_{\succcurlyeq \mathcal{O}})^!"']
\end{tikzcd}.$$

Proof. By [3, Lemma 3.1.7], we need to check that the following lax-commutative diagram commutes:

$$\begin{tikzcd}
\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}} \arrow[r, hook, "F_{\mathcal{O}}"] & D\left(\frac{LG_{\succcurlyeq \mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right) \\
\mathrm{Tr}(\Theta, \mathcal{H}) \arrow[r, hook, "F"'] \arrow[u, "\pi_{\mathrm{Tr},\mathcal{O}}"] \arrow[ur, phantom, "\Leftarrow"] & D\left(\frac{LG}{\mathrm{Ad}_\theta(LG)}\right) \arrow[u, "(i_{\succcurlyeq \mathcal{O}})^!"']
\end{tikzcd}.$$

Concretely, given $\mathscr{F} \in \mathrm{Tr}(\Theta, \mathcal{H})$ we need to check that $(i_{\succcurlyeq \mathcal{O}})^!(\mathscr{F}) \in \mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}}$, that is $(i_{\succcurlyeq \mathcal{O}})_! \circ (i_{\succcurlyeq \mathcal{O}})^!(\mathscr{F}) \in \mathrm{Tr}(\Theta, \mathcal{H})$. By (1.31) this is equivalent to checking that $(p_{\mathcal{O}})_* \circ p_{\mathcal{O}}^*(\mathscr{F}) \in \mathrm{Tr}(\Theta, \mathcal{H})$.

For $\mathcal{O}' \not\succcurlyeq \mathcal{O}$, let $j_{\mathcal{O}',\mathcal{O}} : \frac{LG_{\mathcal{O}'}}{\mathrm{Ad}_\theta(LG)} \hookrightarrow \frac{Z_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}$ be the qcqs open stratum and $k_{\mathcal{O}',\mathcal{O}} : \frac{Z_{\mathcal{O}\setminus\mathcal{O}'}}{\mathrm{Ad}_\theta(LG)} := \frac{Z_{\mathcal{O}} \setminus LG_{\mathcal{O}'}}{\mathrm{Ad}_\theta(LG)} \hookrightarrow \frac{Z_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}$ its fp-closed complement. For any $\mathscr{F} \in D\left(\frac{Z_{\mathcal{O}}}{\mathrm{Ad}_\theta(LG)}\right)$ we have a cofiber-fiber sequence

$$(k_{\mathcal{O}',\mathcal{O}})_* \circ (k_{\mathcal{O}',\mathcal{O}})^!(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow (\jmath_{\mathcal{O}',\mathcal{O}})_* \circ (\jmath_{\mathcal{O}',\mathcal{O}})^!(\mathscr{F}).$$

Since $\frac{Z_{\mathcal{O}\setminus\mathcal{O}'}}{\mathrm{Ad}_\theta(LG)} = \bigsqcup_{\mathcal{O}'' \not\succcurlyeq \mathcal{O}\, \mathcal{O}'' \neq \mathcal{O}'} \frac{LG_{\mathcal{O}''}}{\mathrm{Ad}_\theta(LG)}$, by induction it is enough to check that:

$$(i_{\mathcal{O}'})_* \circ (i_{\mathcal{O}'})^*(\mathscr{F}) \text{ and } (i_{\mathcal{O}'})_! \circ (i_{\mathcal{O}'})^*(\mathscr{F}) \in \mathrm{Tr}(\Theta, \mathcal{H}), \quad \text{for every } \mathcal{O}' \not\succcurlyeq \mathcal{O}. \tag{4.19}$$

Recall that $\mathscr{F} \in \mathrm{Tr}(\Theta, \mathcal{H})$ if it can be written as a colimit of $q_* \circ p^!(\mathscr{G})$ where $\mathscr{G} \in \mathcal{H}$. Moreover, we can write $\mathrm{colim}_{\breve{W}} (f_{\overline{w}})_* \circ f_{\overline{w}}^!(\mathscr{G}) \xrightarrow{\sim} \mathscr{G}$, where $f_w : \mathrm{Iw}\backslash LG_{\leqslant w}/\mathrm{Iw} \hookrightarrow X$. Since any sheaf on $D(\mathrm{Iw}\backslash LG_{\leqslant w}/\mathrm{Iw})$ is a colimit of $\omega_{\mathrm{Iw}\backslash LG_{w'}/\mathrm{Iw}}$ for $w' \leqslant w$, it is enough to consider $\mathscr{G} = \mathscr{F}_w$ as defined in Lemma 4.10. Thus, we need to check that:

$$(i_{\mathcal{O}'})_* \circ i_{\mathcal{O}'}^*(\mathscr{F}_w) \text{ and } (i_{\mathcal{O}'})_! \circ i_{\mathcal{O}'}^*(\mathscr{F}_w) \in \mathrm{Tr}(\Theta, \mathcal{H}), \text{ for every } \mathcal{O}' \not\succcurlyeq \mathcal{O} \text{ and } w \in \breve{W}. \tag{4.20}$$

By Lemma 4.10, we can assume that w is of minimal length in its θ-conjugacy class $\mathcal{O}_w$. In this case, by §2.2.1 LG_w is contained in a single Newton stratum of LG. Thus, q_w factors as the following Cartesian diagram:

$$\begin{tikzcd}
\frac{LG_w}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[r, "\imath^{\breve{I}}_{w,\mathcal{O}_w}"] & \frac{LG_{\mathcal{O}_w}}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[r, "\imath^{\breve{I}}_{\mathcal{O}_w}"] \arrow[d, "q'_{\mathcal{O}_w}"'] & \frac{LG}{\mathrm{Ad}_\theta(\mathrm{Iw})} \arrow[d, "q"] \\
& \frac{LG_{\mathcal{O}_w}}{\mathrm{Ad}_\theta(LG)} \arrow[r, "\imath_{\mathcal{O}_w}"'] & \frac{LG}{\mathrm{Ad}_\theta(LG)}
\end{tikzcd} .$$

Without loss of generality, we can assume that $\mathcal{O}_w \not\succeq \mathcal{O}$. Thus, we obtain:

$$\begin{aligned}
(q_w)_* \circ p_w^!(\omega_{\mathrm{Iw}\backslash LG_w/\mathrm{Iw}}) &\simeq (\imath_{\mathcal{O}_w})_* \circ (q'_{\mathcal{O}_w})_* \circ (\imath^{\breve{I}}_{\mathcal{O}_w})^* \circ (\imath^{\breve{I}}_{\mathcal{O}_w})_* \circ (\imath^{\breve{I}}_{w,\mathcal{O}_w})_* \circ p_w^!(\omega_{\mathrm{Iw}\backslash LG_w/\mathrm{Iw}}) \\
&\simeq (\imath_{\mathcal{O}_w})_* \circ (\imath_{\mathcal{O}_w})^* \circ q_* \circ (\imath^{\breve{I}}_{\mathcal{O}_w})_* \circ (\imath^{\breve{I}}_{w,\mathcal{O}_w})_* \circ p_w^!(\omega_{\mathrm{Iw}\backslash LG_w/\mathrm{Iw}}) \\
&\simeq (\imath_{\mathcal{O}_w})_* \circ (\imath_{\mathcal{O}_w})^* \circ (q_w)_* \circ p_w^!(\omega_{\mathrm{Iw}\backslash LG_w/\mathrm{Iw}}),
\end{aligned}$$

where we used Lemma 1.17 ((v)) for the base change isomorphism in the second line. A similar computation gives $(\imath_{\mathcal{O}_w})_! \circ \imath^*_{\mathcal{O}_w} \circ (q_w)_* \circ p_w^!(\omega_{\mathrm{Iw}\backslash LG_w/\mathrm{Iw}}) \simeq (q_w)_* \circ p_w^!(\omega_{\mathrm{Iw}\backslash LG_w/\mathrm{Iw}})$, because q, q_w, and $q'_{\mathcal{O}_w}$ are ind-fp proper. So (4.20) trivially holds when w is of minimal length. This finishes the proof.

□

4.6. **Proof of Theorem 4.3.** (1) By Lemmas 4.4, 4.5, and 4.6 the augmented simplicial object $\mathcal{H}_\bullet : \Delta_+^{\mathrm{op}} \to \mathrm{Lincat}_E$ satisfies the Beck–Chevalley condition, so Proposition 4.1 implies that $\mathrm{Tr}(\Theta, \mathcal{H}) \to D\left(\frac{\breve{G}}{\mathrm{Ad}_\theta(\breve{G})}\right)$ is fully faithful. The last claim is clear.

(2) By Proposition 4.1, $\mathrm{Tr}(\Theta, \mathcal{H})$ is generated under colimits by the essential image of CH. Notice that the definition of $\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}}$ is equivalent to the formulations in (4.13) and (4.14). By Lemma 4.9 and Lemma 4.11 the category $\mathrm{Tr}(\Theta, \mathcal{H})_{\succeq \mathcal{O}}$ satisfy the conditions of Proposition 4.8. Thus, we get an open $\breve{W} /\!\!/_\theta \breve{W}$-stratification of $\mathrm{Tr}(\Theta, \mathcal{H})$.

(3) The claim follows from Lemma 4.10 and a similar argument as in the proof of Lemma 4.9.

Appendix A. Stratifications of categories

A.1. **Recollement.** Let $\mathcal{X}$ be a presentable stable ∞-category. We say a subcategory $\jmath : \mathcal{U} \hookrightarrow \mathcal{X}$ is *open*, resp. $\imath : \mathcal{Z} \hookrightarrow \mathcal{X}$ is *closed*, if these inclusions extend to diagrams:

$$\begin{tikzcd}
\mathcal{U} \arrow[r, bend left, "\jmath"] \arrow[r, bend right, dashed, "\jmath^{\mathrm{RR}}"'] & \mathcal{X} \arrow[l, dashed, "\jmath^{\mathrm{R}}"']
\end{tikzcd}, \quad \text{resp.} \quad
\begin{tikzcd}
\mathcal{Z} \arrow[r, "\imath"] & \mathcal{X} \arrow[l, bend right, dashed, "\imath^{\mathrm{L}}"'] \arrow[l, bend left, dashed, "\imath^{\mathrm{R}}"]
\end{tikzcd} .$$

We follow the convention of [25, §A.1.1] for the terminology of closed and open (see *loc. cit.* for a remark on how this compares to the different convention of [3]). Also, notice that in [1, §3.1] a closed subcategory is referred to as *admissible*.

Let $\mathrm{Op}_{\mathcal{X}} \hookrightarrow (\mathrm{Pr}^{\mathrm{L}})_{/\mathcal{X}}$ (resp. $\mathrm{Cl}_{\mathcal{X}}$) be the full subcategory of open (resp. closed) subcategories of $\mathcal{X}$. A *recollement* is a diagram:

(A.1)
$$\begin{tikzcd}
\mathcal{U} \arrow[r, bend left, "\jmath"] \arrow[r, bend right, "\jmath^{\mathrm{RR}}"'] & \mathcal{X} \arrow[l, "\jmath^{\mathrm{R}}"'] \arrow[r, bend left, "\imath_{\mathrm{L}}"] \arrow[r, bend right, "\imath_{\mathrm{R}}"'] & \mathcal{Z} \arrow[l, "\imath"']
\end{tikzcd} ,$$

such that $\operatorname{Im} \jmath = \ker \imath_{\mathrm{L}}$, $\operatorname{Im} \imath = \ker \jmath^{\mathrm{R}}$, and $\operatorname{Im} \jmath^{\mathrm{RR}} = \ker \imath^{\mathrm{R}}$. In the diagram (A.1) either the sub-diagram between $\mathcal{U}$ and $\mathcal{X}$, or $\mathcal{Z}$ and $\mathcal{X}$, determines the rest of the diagram. More precisely, we have:

Proposition A.1. *[1, Proposition 3.4] Let $\imath : \mathcal{Y} \hookrightarrow \mathcal{X}$ denote a subcategory of $\mathcal{X}$, then:*

(1) if $\mathcal{Y}$ is open then one has a recollement:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, bend left, "\imath"] \arrow[r, bend right, "\imath^{\mathrm{RR}}"'] & \mathcal{X} \arrow[l, "\imath^{\mathrm{R}}"'] \arrow[r, bend left, "\imath^{\perp,\mathrm{L}}"] \arrow[r, bend right, "\imath^{\perp,\mathrm{R}}"'] & \mathcal{Y}^{\perp} \arrow[l, "\imath^{\perp}"'] \end{tikzcd} , \tag{A.2}$$

where $\mathcal{Y}^{\perp} := \{x \in \mathcal{X} \mid \mathrm{Map}_{\mathcal{X}}(\imath(y), x) = 0, \text{ for all } y \in \mathcal{Y}\}$.

(2) if $\mathcal{Y}$ is closed, then one has a recollement:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "\imath"] & \mathcal{X} \arrow[l, bend right, "\imath^{\mathrm{L}}"'] \arrow[l, bend left, "\imath^{\mathrm{R}}"] \arrow[r, "{}^{\perp}\imath^{\mathrm{R}}"] & {}^{\perp}\mathcal{Y} \arrow[l, bend right, "{}^{\perp}\imath"'] \arrow[l, bend left, "{}^{\perp}\imath^{\mathrm{RR}}"] \end{tikzcd} , \tag{A.3}$$

where ${}^{\perp}\mathcal{Y} := \{x \in \mathcal{X} \mid \mathrm{Map}_{\mathcal{X}}(x, \imath(y)) = 0, \text{ for all } y \in \mathcal{Y}\}$.

Remark A.2. Given a continuous functor $F : \mathcal{Z} \to \mathcal{X}$ between presentable stable ∞-categories, then F admits a right adjoint F^{R}. Moreover, if F preserves compact objects, then F^{R} preserves filtered colimits. Thus, by the adjoint functor theorem, F^{R} admits a further right adjoint $F^{\mathrm{R,R}}$.

A.2. **Stratification of presentable stable ∞-categories.** Let P be a partially ordered set (in short, poset). For some results, we need to assume that P is *down-finite*, i.e. for every $p \in \mathsf{P}$ the subset $\mathsf{P}_{\leqslant p} \subseteq \mathsf{P}$ is finite.

Definition A.3. [3, Definition 1.3.2] An *open* P*-stratification* of $\mathcal{X}$ is a functor

$$\mathcal{U}_{\bullet} : \mathsf{P} \longrightarrow \mathrm{Op}_{\mathcal{X}}, \quad p \longmapsto \begin{tikzcd} \mathcal{U}_p \arrow[r, hook, bend left, "\jmath_p"] \arrow[r, bend right, "\jmath_p^{\mathrm{RR}}"'] & \mathcal{X} \arrow[l, "\jmath_p^{\mathrm{R}}"'] \end{tikzcd}$$

satisfying:

(i) $\operatorname{colim}_{p \in P} \mathcal{U}_p \simeq \mathcal{X}$ (in Pr^{L});

(ii) for every $p, q \in \mathsf{P}$ one has a factorization:

$$\begin{tikzcd} \mathcal{U}_p \cap \mathcal{U}_q \arrow[r, hook] & \mathcal{U}_q \\ \mathcal{U}_p \arrow[u, dashed] \arrow[r, hook, "\jmath_p"'] & \mathcal{X}, \arrow[u, "\jmath_q^{\mathrm{R}}"'] \end{tikzcd} ,$$

where $\mathcal{U}_p \cap \mathcal{U}_q$ is the pullback of $\mathcal{U}_p \xrightarrow{\jmath_p} \mathcal{X} \xleftarrow{\jmath_q} \mathcal{U}_q$;

(iii) $\mathcal{U}_p \cap \mathcal{U}_q = \bigcup_{(\leqslant p) \cap (\leqslant q)} \mathcal{U}_r := \operatorname*{colim}_{\mathsf{P}_{\leqslant p} \cap \mathsf{P}_{\leqslant q}} \mathcal{U}_r$ (in $(\mathrm{Pr}^{\mathrm{L}})_{/\mathcal{X}}$) for every $p, q \in \mathsf{P}$.

Definition A.4. A *closed* P-*stratification* of $\mathcal{X}$ is a (strict) functor:

$$\begin{tikzcd} \mathcal{Z}_\bullet : \mathsf{P} \longrightarrow \mathrm{Cl}_{\mathcal{X}}, \quad p \longmapsto & \mathcal{Z}_p \arrow[r, hook, "\imath_p" description] & \mathcal{X} \arrow[l, bend right, "\imath_p^{\mathrm{L}}"'] \arrow[l, bend left, "\imath_p^{\mathrm{R}}"] \end{tikzcd}$$

satisfying:

(i) $\mathrm{colim}_{p \in \mathsf{P}} \mathcal{Z}_p \simeq \mathcal{X}$ (in Pr^{L});

(ii) for every p, q one has a factorization:

$$\begin{tikzcd} \mathcal{Z}_p \cap \mathcal{Z}_q \arrow[r, hook] & \mathcal{Z}_q \\ \mathcal{Z}_p \arrow[u, dashed] \arrow[r, hook, "\imath_p"'] & \mathcal{X}, \arrow[u, "\imath_q^{\mathrm{L}}"'] \end{tikzcd}$$

where $\mathcal{Z}_p \cap \mathcal{Z}_q$ is the pullback of $\mathcal{Z}_p \xrightarrow{\imath_p} \mathcal{X} \xleftarrow{\imath_q} \mathcal{Z}_q$;

(iii) $\mathcal{Z}_p \cap \mathcal{Z}_q = \bigcup_{(\leqslant p) \cap (\leqslant q)} \mathcal{Z}_r := \mathrm{colim}_{\mathsf{P}_{\leqslant p} \cap \mathsf{P}_{\leqslant q}} \mathcal{Z}_r$ (in $(\mathrm{Pr}^{\mathrm{R}})_{/\mathcal{X}}$).

Notice that condition (iii) for the open stratification considers the colimit in Pr^{L} whereas for the closed stratification one takes the colimit in Pr^{R}. When P is totally ordered, conditions (ii) and (iii) are automatic, thus Definition A.4 recovers [1, Definition 3.6].

Remark A.5. Definition A.4 should be seem as a categorical analogue of the decomposition of a scheme (or any more general geometric object) $X \simeq \cup_I Z_i$ where each $Z_i \hookrightarrow X$ is closed and $Z_i \cap Z_j = \cup_{k \leqslant i \text{ and } k \leqslant j} Z_k$. Definition A.3 is the categorical analogue of decomposing $X \simeq \cup_{I^{\mathrm{op}}} U_i$ where each $U_i \hookrightarrow X$ is open and $U_i \cap U_j = \cup_{k \geqslant i \text{ and } k \geqslant j} U_k$. Notice that in nice situations we can pass from the data of $\{Z_i\}_{i \in I}$ to $\{U_i\}_{i \in I^{\mathrm{op}}}$, via $U_i := X \backslash Z_i$ and they both are equivalent to giving a decomposition $X \simeq \sqcup_I X_i$, where each X_i is locally closed with $\overline{X_i} = \sqcup_{j \leqslant i} X_j$ (the categorical version is given in §A.5). Notice that this is a more structure situation that simply saying that for each $i \in I$ one has an open-closed decomposition $Z_i \hookrightarrow X \hookleftarrow U_i$ of X.

A.3. **Open-closed correspondence.** We start by explaining a construction that allows us to pass from closed to open stratifications and vice-versa.

Let $\mathcal{Z}_\bullet : \mathsf{P} \to \mathrm{Cl}_{\mathcal{X}}$ be a closed stratification. For each $p \in P$ let $\mathsf{R}(\mathcal{Z})_p$ be defined by the following recollement diagram:

$$\begin{tikzcd} \mathsf{R}(\mathcal{Z})_p \arrow[r, bend left, hook, "\jmath_p"] \arrow[r, bend right, hook, "\jmath_p^{\mathrm{RR}}"'] & \mathcal{X} \arrow[l, "\jmath_p^{\mathrm{R}}" description] \arrow[r, bend left, "\imath_p^{\mathrm{L}}"] \arrow[r, bend right, "\imath_p^{\mathrm{R}}"'] & \bigcup_{q \not\geqslant p} \mathcal{Z}_q \arrow[l, hook', "\imath_p" description] \end{tikzcd} ,$$

where we claim that $\cup_{q \not\geqslant p} \mathcal{Z}_q = \mathrm{colim}_{\mathsf{P}_{\not\geqslant p}} \mathcal{Z}_q \in \mathrm{Cl}_{\mathcal{X}}$. Indeed, via the equivalence $\mathrm{Pr}^{\mathrm{R}} \simeq (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}}$ we have:

$$\mathrm{colim}_{\mathsf{P}_{\not\geqslant p}} \mathcal{Z}_q \xrightarrow{\sim} \lim_{(\mathsf{P}_{\not\geqslant p})^{\mathrm{op}}} \mathcal{Z}_q$$

where the limit is computed in Pr^{L}, equivalently in Pr (cf. [3, Observation 2.3.9]).

Concretely, by Proposition A.1 we have $\mathsf{R}(\mathcal{Z})_p \simeq (\bigcup_{q\not\geqslant p} \mathcal{Z}_q)^\perp \simeq \bigcap_{q\not\geqslant p} \mathcal{Z}_q^\perp$. Given $p_1 \to p_2$, notice that the diagram:

$$\begin{tikzcd}
\mathcal{Z}_{p_1} \arrow[r, "\imath_{p_1,p_2}"] \arrow[d, "\imath_{p_1}"'] & \mathcal{Z}_{p_2} \arrow[d, "\imath_{p_2}"] \\
\mathcal{X} \arrow[r, "\mathrm{id}_{\mathcal{X}}"'] & \mathcal{X}
\end{tikzcd}$$

is horizontally left and right adjointable. Thus, the following commutative diagram defines $\jmath_{p_2,p_1}$:

$$\begin{tikzcd}
\bigcap\limits_{q\not\geqslant p_1} \mathcal{Z}_q^\perp \arrow[r, hook, "\jmath_{p_1}"] & \mathcal{X} \arrow[r, "\imath^{\mathrm{L}}_{\not\geqslant p_1}"] & \bigcup\limits_{q\not\geqslant p_1} \mathcal{Z}_q \\
\bigcap\limits_{q\not\geqslant p_2} \mathcal{Z}_q^\perp \arrow[r, hook, "\jmath_{p_2}"'] \arrow[u, hook, dashed, "\jmath_{p_2,p_1}"] & \mathcal{X} \arrow[r, "\imath^{\mathrm{L}}_{\not\geqslant p_2}"'] \arrow[u, "\mathrm{id}_{\mathcal{X}}"] & \bigcup\limits_{q\not\geqslant p_2} \mathcal{Z}_q \arrow[u, "(\imath_{\not\geqslant p_1,\not\geqslant p_2})^{\mathrm{L}}"']
\end{tikzcd}.$$

Moreover, notice that $(\jmath_{p_2,p_1})^{\mathrm{R}} := (\jmath_{p_1})^{\mathrm{R}} \circ \jmath_{p_2}$ gives a right adjoint to $\jmath_{p_2,p_1}$ and that $(\jmath_{p_2,p_1})^{\mathrm{RR}} := (\jmath_{p_1})^{\mathrm{R}} \circ (\jmath_{p_2})^{\mathrm{RR}}$ is a further right adjoint. Thus, we obtain a functor:

$$\mathsf{R}(\mathcal{Z})_\bullet : \mathsf{P}^{\mathrm{op}} \longrightarrow \mathrm{Op}_{\mathcal{X}}.$$

Similarly, given $\mathcal{U}_\bullet : \mathsf{P} \to \mathrm{Op}_{\mathcal{X}}$ an open stratification, we can define $\check{\mathsf{R}}(\mathcal{U})_p$ via the recollement diagram:

$$\begin{tikzcd}
\bigcup\limits_{q\not\geqslant p} \mathcal{U}_q \arrow[r, bend left, hook, "\jmath_p"] & \mathcal{X} \arrow[l, "\jmath_p^{\mathrm{R}}"'] \arrow[r, bend left, "\imath_p^{\mathrm{L}}"] \arrow[r, bend right, "\imath_p^{\mathrm{R}}"'] & \check{\mathsf{R}}(\mathcal{U})_p \arrow[l, hook', "\imath_p"'] \\
\arrow[from=1-1, to=1-2, bend right, hook', "\jmath_p^{\mathrm{RR}}"']
\end{tikzcd}.$$

The same argument as above gives that we have a functor $\check{\mathsf{R}}(\mathcal{U})_\bullet : \mathsf{P}^{\mathrm{op}} \to \mathrm{Cl}_{\mathcal{X}}$.

Proposition A.6. *Let P be a down-finite poset and $\mathcal{X}$ a presentable stable ∞-category. The following data are equivalent:*

(1) a closed P-stratification $\mathcal{Z}_\bullet : \mathsf{P} \to \mathcal{X}$;

(2) an open P^{op}-stratification $\mathcal{U}_\bullet : \mathsf{P}^{\mathrm{op}} \to \mathcal{X}$.

Proof. We prove (1) $\Rightarrow$ (2), and the direction (2) $\Rightarrow$ (1) is similar. We will prove that $\mathsf{R}(\mathcal{Z})_\bullet : \mathsf{P}^{\mathrm{op}} \to \mathrm{Op}_{\mathcal{X}}$ is an open stratification. First notice that one has:

$$\mathcal{X}/\operatorname*{colim}_{\mathsf{P}^{\mathrm{op}}} \mathsf{R}(\mathcal{Z})_\bullet \simeq \operatorname*{colim}_{p\in\mathsf{P}^{\mathrm{op}}} \bigcup_{q\not\geqslant p} \mathcal{Z}_q \simeq 0$$

since the indexing set $\mathsf{P}_{\not\geqslant p}$ vanishes for $p \in \mathsf{P}^{\mathrm{op}}$ a maximal element, which exists if P is down-finite.

We verify conditions (ii) and (iii) from Definition A.3.

Condition (iii) follows directly from the computation:

$$\operatorname*{colim}_{q\in(\mathsf{P}_{\leqslant p_1})^{\mathrm{op}}\cap(\mathsf{P}_{\leqslant p_2})^{\mathrm{op}}} \mathsf{R}(\mathcal{Z})_q \simeq \lim_{q\in\mathsf{P}_{\leqslant p_1}\cap\mathsf{P}_{\leqslant p_2}} \bigcap_{r\in\mathsf{P}_{\not\geqslant q}} \mathcal{Z}_q^\perp \simeq \bigcap_{r\in\mathsf{P}_{\not\geqslant p_1}\cap\mathsf{P}_{\not\geqslant p_2}} \mathcal{Z}_r^\perp \simeq \mathsf{R}(\mathcal{Z})_{p_1} \cap \mathsf{R}(\mathcal{Z})_{p_2}.$$

For (ii), notice that the diagram on the left determines the diagram on the right by considering quotients inside of $\mathcal{X}$:

$$
\begin{tikzcd}
\bigcup\limits_{(\not\geq p_1)\cap(\not\geq p_2)} \mathcal{Z}_r & \bigcup\limits_{(\not\geq p_2)} \mathcal{Z}_r \arrow[l, "(\imath_{\not\geq p_1\cap\not\geq p_2,\not\geq p_2})^{\mathrm{L}}"'] \\
\bigcup\limits_{(\not\geq p_1)} \mathcal{Z}_r \arrow[u, "(\imath_{\not\geq p_1\cap\not\geq p_2,\not\geq p_1})^{\mathrm{L}}"] & \mathcal{X} \arrow[l, "(\imath_{\not\geq p_1})^{\mathrm{L}}"] \arrow[u, "(\imath_{\not\geq p_2})^{\mathrm{L}}"']
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathsf{R}(\mathcal{Z})_{p_1} \cap \mathsf{R}(\mathcal{Z})_{p_2} \arrow[r, hook, "\jmath_{p_1\cap p_2,p_2}"] \arrow[d, hook, "\jmath_{p_1\cap p_2,p_1}"'] & \mathsf{R}(\mathcal{Z})_{p_2} \arrow[d, hook, "\jmath_{p_2}"] \\
\mathsf{R}(\mathcal{Z})_{p_1} \arrow[r, hook, "\jmath_{p_1}"'] & \mathcal{X}
\end{tikzcd} .
$$

By passing to the right adjoint on the horizontal arrows on both diagrams we obtain lax-commutative diagrams:

$$
\begin{tikzcd}
\bigcup\limits_{(\not\geq p_1)\cap(\not\geq p_2)} \mathcal{Z}_r \arrow[r, "\imath_{\not\geq p_1\cap\not\geq p_2,\not\geq p_2}"] \arrow[dr, phantom, "\Rightarrow"] & \bigcup\limits_{(\not\geq p_2)} \mathcal{Z}_r \\
\bigcup\limits_{(\not\geq p_1)} \mathcal{Z}_r \arrow[u, "(\imath_{\not\geq p_1\cap\not\geq p_2,\not\geq p_1})^{\mathrm{L}}"] \arrow[r, "\imath_{\not\geq p_1}"'] & \mathcal{X} \arrow[u, "(\imath_{\not\geq p_2})^{\mathrm{L}}"']
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathsf{R}(\mathcal{Z})_{p_1} \cap \mathsf{R}(\mathcal{Z})_{p_2} \arrow[d, hook, "\jmath_{p_1\cap p_2,p_1}"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{R}(\mathcal{Z})_{p_2} \arrow[l, "(\jmath_{p_1\cap p_2,p_2})^{\mathrm{R}}"'] \arrow[d, hook, "\jmath_{p_2}"] \\
\mathsf{R}(\mathcal{Z})_{p_1} & \mathcal{X} \arrow[l, "(\jmath_{p_1})^{\mathrm{R}}"]
\end{tikzcd} ,
$$

where the double arrow specifies the not necessarily invertible 2-morphism. By a closed variant of [3, Lemma 3.4.5] the left diagram actually commutes, which implies that the right diagram also commutes. $\square$

A.4. **Glueing diagrams.**

A.4.1. *Open variant.* Let $\mathcal{U}_\bullet : \mathsf{P} \to \mathrm{Op}_{\mathcal{X}}$ be an open stratification of $\mathcal{X}$. For each $p \in \mathsf{P}$ we define the *pth stratum* $\mathcal{X}_p$ by the following recollement diagram:

$$
\begin{tikzcd}
\operatorname*{colim}\limits_{r\in\mathsf{P}_{\not\geq q}} \mathcal{U}_r \arrow[r, bend left, "\jmath^{\mathrm{L}}_{\not\geq q,q}"] \arrow[r, bend right, "\jmath^{\mathrm{R}}_{\not\geq q,q}"'] & \mathcal{U}_p \arrow[l, "\jmath_{\not\geq q,q}"'] \arrow[r, bend left, "\mathring{\imath}^{\mathrm{L}}_p"] \arrow[r, bend right, "\mathring{\imath}^{\mathrm{R}}_p"'] & \mathcal{X}_p \arrow[l, hook', "\mathring{\imath}_p"']
\end{tikzcd} .
$$

We then define:

$$
\Phi_p : \mathcal{X} \xrightarrow{\jmath^{\mathrm{R}}_p} \mathcal{U}_p \xrightarrow{\mathring{\imath}^{\mathrm{L}}_p} \mathcal{X}_p, \quad\text{and}\quad \rho^p : \mathcal{X}_p \xrightarrow{\mathring{\imath}_p} \mathcal{U}_p \xrightarrow{\jmath^{\mathrm{RR}}_p} \mathcal{X},
$$

which give an adjunction (Φ_p, ρ^p). For each $p \to q$ in P we define the *open glueing functor* to be:

$$
\mathring{\Gamma}^p_q : \mathcal{X}_p \xhookrightarrow{\rho^p} \mathcal{X} \xrightarrow{\Phi_q} \mathcal{X}_q \ . \tag{A.4}
$$

Notice that for each pair of composable arrows $p \to q \to r$ in P we obtain a natural morphism: $\mathring{\Gamma}^p_r \Rightarrow \mathring{\Gamma}^q_r \circ \mathring{\Gamma}^p_q$. These assemble into a left-lax P-module (see [3, §A.1] for a precise definition):

$$
\mathsf{P} \xrightarrow[\text{l.lax}]{\mathring{\mathscr{G}}} \mathrm{Pr}^{\mathrm{L}}. \tag{A.5}
$$

We refer to (A.5) as the *open glueing diagram*.

A.4.2. *Closed variant.* Let $\mathcal{Z}_\bullet : \mathsf{P} \to \mathrm{Cl}_{\mathcal{X}}$ be a closed stratification of $\mathcal{X}$. Following [3, Definition 1.3.5] we pose:

Definition A.7. For each $p \in \mathsf{P}$ the *pth stratum* $\mathcal{X}_p$ of $\mathcal{Z}_\bullet$ is defined by the following recollement diagram:

$$\begin{tikzcd} \mathcal{X}_p & \mathcal{Z}_p \arrow[l, "\pi_p"'] \arrow[l, bend right, "\pi_p^{\mathrm{L}}"'] \arrow[l, bend left, "\pi_p^{\mathrm{R}}"] & \bigcup_{\mathsf{P}_{\not\geqslant p}} \mathcal{Z}_r \arrow[l, "\imath_{\not\geqslant p,p}"'] \end{tikzcd} ,$$

where $\bigcup_{\mathsf{P}_{\not\geqslant p}} \mathcal{Z}_r := \operatorname{colim}_{r \in \mathsf{P}_{\not\geqslant p}} \mathcal{Z}_r \simeq \lim_{r \in (\mathsf{P}_{\not\geqslant p})^{\mathrm{op}}} \mathcal{Z}_r$, where the limit is taken in Pr^{L} with respect to $\imath^{\mathrm{L}}_{p,q} : \mathcal{Z}_q \to \mathcal{Z}_p$ for $p \to q$ in P.

Now we define:

$$\Psi^p : \mathcal{X} \xrightarrow{\imath^{\mathrm{L}}_p} \mathcal{Z}_p \xrightarrow{\pi_p} \mathcal{X}_p, \quad \text{and} \quad \nu_p : \mathcal{X}_p \xrightarrow{\pi^{\mathrm{R}}_p} \mathcal{Z}_p \xrightarrow{\imath_p} \mathcal{X},$$

which give an adjunction (Ψ^p, ν_p). For each $p \to q$ we define the *closed glueing functor*:

$$\Gamma^p_q : \mathcal{X}_p \xhookrightarrow{\Psi^p} \mathcal{X} \xrightarrow{\nu_q} \mathcal{X}_q.$$

Notice that for each composable pair of arrows $p \to q \to r$, the counit of the adjunction (Ψ_q, ν^q) gives a natural morphism $\Gamma^q_p \circ \Gamma^r_q \Rightarrow \Gamma^r_p$.

Remark A.8. Notice that if the bottom line composition vanishes, then one has a factorization:

$$\begin{tikzcd} & \bigcup_{\mathsf{P}_{\leqslant p,\leqslant q}} \mathcal{Z}_r / \bigcup_{\mathsf{P}_{\leqslant p,\leqslant q,\geq p}} \mathcal{Z}_r \arrow[r] & \mathcal{X}_q \\ \bigcup_{\mathsf{P}_{\not\geqslant p,\leqslant q}} \mathcal{Z}_r \arrow[r, hook] & \bigcup_{\mathsf{P}_{\leqslant p,\leqslant q}} \mathcal{Z}_r \arrow[r, hook] \arrow[u] & \mathcal{Z}_q \arrow[u, "\pi_q"'] \\ \bigcup_{\mathsf{P}_{\not\geqslant p}} \mathcal{Z}_r \arrow[r, hook] \arrow[u] & \mathcal{Z}_p \arrow[r, hook, "\imath_p"'] \arrow[u] & \mathcal{X} \arrow[u, "\imath^{\mathrm{R}}_q"'] \end{tikzcd} ,$$

here we abbreviated $\mathsf{P}_{(\leqslant p) \cap (\leqslant q)} = \mathsf{P}_{\leqslant p, \leqslant q}$ and similarly for other subscripts. Notice that if $p \not\leqslant q$ then $\mathsf{P}_{\leqslant p,\leqslant q} = \mathsf{P}_{\leqslant p,\leqslant q,\geqslant p}$, so $\bigcup_{\mathsf{P}_{\leqslant p,\leqslant q}} \mathcal{Z}_r / \bigcup_{\mathsf{P}_{\leqslant p,\leqslant q,\geqslant p}} \mathcal{Z}_r \simeq 0$ and the composite $\nu_q \circ \imath_p$ vanishes. In particular, Γ^p_q also vanishes. The open glueing functors have a similar property, namely $\mathring{\Gamma}^p_q$ vanishes whenever $q \not\leqslant p$.

The glueing functors assemble into a right-lax P^{op}-module:

$$\mathsf{P}^{\mathrm{op}} \xrightarrow[\mathrm{r.lax}]{\mathscr{G}} \mathrm{Pr}^{\mathrm{L}}. \tag{A.6}$$

We refer to (A.6) as the *closed glueing diagram.*

The main result of [3] is that one can reconstruct an (open/closed) stratification from an (open/closed) glueing diagram.

Theorem A.9. *[3, Theorem A (1) and (2)] Assume that* P *is down-finite. Let* $\mathcal{X}$ *be a presentable stable* ∞*-category.*

(1) The data of an open P*-stratification of* $\mathscr{X}$ *is equivalent to the data of (A.5). Moreover, one has an equivalence:*

$$\mathscr{X} \xrightarrow{\simeq} \lim_{\mathrm{sd}(\mathsf{P})} \mathscr{G}^{\mathrm{op}}(\mathscr{X}),$$

where $\mathrm{sd}(\mathsf{P})$ *is the subdivision poset associated to* P *(see [3, Definition A.4.2]).*

(2) The data of a closed P*-stratification of* $\mathscr{X}$ *is equivalent to the data of (A.6). Moreover, one has an equivalence:*

$$\operatorname*{colim}_{\mathrm{sd}(\mathsf{P})} \mathscr{G}(\mathscr{X}) \xrightarrow{\simeq} \mathscr{X}.$$

Remark A.10. Let $\mathscr{Z}_\bullet : \mathsf{P} \to \mathscr{X}$ be a closed stratification and $\mathsf{R}(\mathscr{Z})_\bullet : \mathsf{P}^{\mathrm{op}} \to \mathscr{X}$ the corresponding open stratification constructed in §A.3. By [3, §1.10] (see §3.2.4 for an example) one also has a *reflected open glueing diagram* associated to $\mathsf{R}(\mathscr{Z})_\bullet$:

$$\mathsf{P}^{\mathrm{op}} \xrightarrow[\mathrm{r.lax}]{\check{\mathring{\mathscr{G}}}} \mathrm{Pr}^{\mathrm{L}}. \, .$$

When P is down-finite, one has equivalences:

$$\operatorname*{colim}_{\mathrm{sd}(\mathsf{P}^{\mathrm{op}})^{\mathrm{op}}} \check{\mathring{\mathscr{G}}}(\mathscr{X}) \xrightarrow{\simeq} \mathscr{X} \xleftarrow{\simeq} \operatorname*{colim}_{\mathrm{sd}(\mathsf{P})} \mathscr{G}(\mathscr{X}).$$

In other words, the glueing of $\mathscr{X}$ given by its closed stratification is equivalent to the reflected glueing given by the corresponding open stratification.

For clarity, we summarize the results of this appendix:

$$\begin{tikzcd}
(\text{cl. strat. } \mathscr{Z}_\bullet : \mathsf{P} \to \mathscr{X}) \arrow[rr, leftrightarrow, dashed, "\text{op.-cl. constr.}"] \arrow[d, dashed, "\text{cl. glueing}"'] & & (\text{op. strat. } \mathscr{U}_\bullet : \mathsf{P}^{\mathrm{op}} \to \mathscr{X}) \arrow[dll, dashed, "\text{refl. op. glueing}"] \arrow[d, dotted, "\text{op. glueing}"] \\
\left(\mathscr{G} : \mathsf{P}^{\mathrm{op}} \underset{\mathrm{r.lax}}{\to} \mathrm{Pr}^{\mathrm{L}}\right) & & \left(\mathring{\mathscr{G}} : \mathsf{P}^{\mathrm{op}} \underset{\mathrm{l.lax}}{\to} \mathrm{Pr}^{\mathrm{L}}\right)
\end{tikzcd}$$

FIGURE 1. Dependence of decomposition notions.

In the above diagram, the dashed arrows are equivalences whenever P is down-finite, whereas the dotted arrow is an equivalence when P^{op} is down-finite. One could conceive other constructions to complete the above diagram, we leave that to the reader.

A.5. **Semi-orthogonal decomposition.** Let $\mathscr{X}$ be a presentable stable ∞-category. By definition, a P*-indexed semi-orthogonal decomposition of* $\mathscr{X}$ is a right lax functor:

$$\mathscr{X}_\bullet : \mathsf{P} \xrightarrow[\mathrm{r.lax}]{} \mathrm{Pr}^{\mathrm{L}}$$

satisfying:

(i) for every $q \not\leq p$ one has $\mathscr{X}_p^{\perp} \hookrightarrow \mathscr{X}_q$;

(ii) $\operatorname{colim}_{\mathrm{sd}(\mathsf{P})} \mathscr{X}_p \xrightarrow{\simeq} \mathscr{X}$ (in Pr^{R}).

Both conditions above have very concrete descriptions. Condition (i) is equivalent to saying that for any $x_p \in \mathscr{X}_p$ and $x_q \in \mathscr{X}_q$ we have: $\mathrm{Map}_{\mathscr{X}}(\Psi^q(x_q), \Psi^p(x_p)) = 0$ for $q \not\leq p$. Condition (ii) is equivalent to requiring that $\mathscr{X}$ is the smallest stable subcategory of $\mathscr{X}$ containing the essential image of $\Psi^p : \mathscr{X}_p \hookrightarrow \mathscr{X}$ for all $p \in \mathsf{P}$.

The following is a consequence of Theorem A.9.

Corollary A.11. *Assume that* P *is down-finite. For* $\mathcal{X}$ *a presentable stable* ∞*-category, the following data are equivalent:*

(1) a closed stratification $\mathcal{Z}_\bullet : \mathsf{P} \to \mathrm{Cl}_{\mathcal{X}}$*;*

(2) an open stratification $\mathcal{U}_\bullet : \mathsf{P}^{\mathrm{op}} \to \mathrm{Op}_{\mathcal{X}}$*;*

(3) a semi-orthogonal decomposition $\mathcal{X}_\bullet : \mathsf{P} \underset{\text{r.lax}}{\to} (\mathrm{Pr}^{\mathrm{L}})_{/\mathcal{X}}$.

Following the arguments in [3, §7], one obtains analogues of all the constructions and results in the appendix for $\mathcal{X}$ an idempotent complete, i.e. its underlying homotopy category is idempotent complete, (small) stable ∞-category.

Remark A.12. When P is totally ordered, the equivalence of (1) and (3) in the version of Corollary A.11 for idempotent complete small stable ∞-categories recovers [1, Proposition 3.8]. When P is *not* totally ordered [1, Definition 3.6] considers a semi-orthogonal decomposition the data of $\mathcal{Z} : \mathsf{P} \to \mathrm{Cl}_{\mathcal{X}}$ satisfying only condition (i) of Definition A.4. We stress that for our applications the correct notion is the one considered in Corollary A.11.

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