CATEGORICAL TRACES I: DELIGNE–LUSZTIG THEORY
SPRING 2026 – SIMIS

ARON HELEODORO

This course has the following goals:

(i) Understand the book [DM20] by Digne and Michel. This book covers traditional material on Deligne–Lusztig theory.

(ii) Introduce the necessary technology from higher category theory and homotopical algebra to understand categorical traces and their relation to Deligne–Lusztig theory. This will follow the discussion in [Zhu25, §4.4] and any other sections of [Zhu25] that are necessary as background.

(iii) Discuss other results such as [Ete24; Ete25b; Ete25a], and how they relate to the categories considered in [Zhu25, §4.4].

(iv) Hopefully, say something new about any of the above topics.

What is not in this course? The recent preprint [GRV25] and its upcoming sequels, have a 2-categorical perspective on Deligne–Lusztig theory and can obtain new results in this area. However, this is not in the scope of this course.

Plan

For the part I we will follow the book [DM20]. The plan is to cover the following topics:

(i) Definition of finite groups of Lie type (roughly 20 pages).

(ii) Harish–Chandra theory (roughly 12 pages).

(iii) "Cohomological induction", aka Deligne–Lusztig theory:

- prerequesites in ℓ-adic cohomology (7 pages);
- Deligne–Lusztig induction (11 pages);
- Character formula and other results (13 pages);
- Geometric conjugacy and Lusztig series (35 pages).

(iv) Regular elements, Gelfand–Graev representations, and regular and semisimple elements (29 pages).

For part II we will follow [Zhu25, §4.4]:

(i) Second discussion of ℓ-sheaves and monodromic sheaves ([Zhu25, §10 and §4.1]).

(ii) Rephrasing Deligne–Lusztig induction [Zhu25, §4.4.1] in terms of stacks.

(iii) Introduction of monodromic tilting sheaf and its properties [Zhu25, §4.4.2].

(iv) Deligne–Lusztig induction as a categorical trace [Zhu25, §4.4.3].

(v) Dudas' theorem relating Deligne–Lusztig induction and Gelfand–Graev representations [Zhu25, §4.4.4].

Date: Last updated April 1, 2026.

Tentative Course Schedule

Week	Date	Topic	References
1	Mar. 3	Introduction and Finite groups of Lie type	[DM20, §4]
2	Mar. 10	(B, N)-pairs	[DM20, §4]
3	Mar. 17	Harish–Chandra theory	[DM20, §5]
4	Mar. 24	ℓ-cohomology (first passage)	[DM20, §8]
5	Mar. 31	Deligne–Lusztig induction	[DM20, §9]
6	Apr. 7	The character formula and other results	[DM20, §10]
7	Apr. 14	Geometric conjugacy and Lusztig series	[DM20, §11.1-11.3]
8	Apr. 21	Jordan decomposition	[DM20, §11.4-11.5]
9	Apr. 28	Regular (unipotent) elements	[DM20, §12.1-12.2]
10	May 5	Gelfand–Graev representations	[DM20, §12.3-12.5]
11	May 12	ℓ-adic sheaves and monodromic sheaves	[Zhu25, §10, §4.1]
12	May 19	Deligne–Lusztig induction (second take)	[Zhu25, §4.4.1]
13	May 26	Tilting objects and projective generators	[Zhu25, §4.4.2] [Ete25b]
14	Jun. 2	Background on categorical traces	[BFN10; HL23] [Zhu25, §7]
15	Jun. 9	Deligne–Lusztig induction (third take)	[Zhu25, §4.4.3]
16	Jun. 16	Gelfand–Graev representation (second take)	[Zhu25, §4.4.4]

Actual Course Schedule

Week	Date	Topic	References
1	Mar. 3	Introduction, Review of Algebraic Groups, Solvable and Reductive Groups	[DM20, §1]
2	Mar. 10	Structure theory of reductive groups	[DM20, §2]
3	Mar. 24	(B, N)-pairs, Bruhat decomposition, Parabolic subgroups and Levi subgroups, construction using quasi-closed subsets of the root system	[DM20, §3.1 – 3.4]
4	Mar. 31	Centralizers of semi-simple elements, $\mathbb{F}_p$-structures, Frobenius endomorphisms, Lang–Steinberg Theorem	[DM20, §3.5, 4.1]
5	Apr. 1	F-stable Borel and maximal tori, finite groups of Lie type	[DM20, §4.2 – 4.3]

References

[BFN10] David Ben-Zvi, John Francis, and David Nadler. "Integral Transforms and Drinfeld Centers in Derived Algebraic Geometry". In: *Journal of the American Mathematical Society* 23.4 (2010), pp. 909–966. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-10-00669-7. URL: https://doi-org.proxy2.library.illinois.edu/10.1090/S0894-0347-10-00669-7.

[DM20] François Digne and Jean Michel. *Representations of Finite Groups of Lie Type*. Vol. 95. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2020, pp. vii+257. ISBN: 978-1-108-72262-9 978-1-108-48148-9.

[Ete24] Arnaud Eteve. *Free Monodromic Hecke Categories and Their Categorical Traces*. Version 2. 2024. DOI: 10.48550/ARXIV.2412.13323. URL: https://arxiv.org/abs/2412.13323 (visited on 04/21/2025). Pre-published.

[Ete25a] Arnaud Eteve. *Applications of the Trace Formalism to Deligne-Lusztig Theory*. Version 1. 2025. DOI: 10.48550/ARXIV.2501.04113. URL: https://arxiv.org/abs/2501.04113 (visited on 04/21/2025). Pre-published.

[Ete25b] Arnaud Eteve. *Tilting Representations of Finite Groups of Lie Type*. Oct. 9, 2025. DOI: 10.48550/arXiv.2412.13326. arXiv: 2412.13326. URL: http://arxiv.org/abs/2412.13326 (visited on 12/17/2025). Pre-published.

[GRV25] Dennis Gaitsgory, Nick Rozenblyum, and Yakov Varshavsky. *Applications of (Higher) Categorical Trace I: The Definition of AGCat*. Dec. 25, 2025. DOI: 10.48550/arXiv.2512.21726. arXiv: 2512.21726. URL: http://arxiv.org/abs/2512.21726 (visited on 01/07/2026). Pre-published.

[HL23] Quoc P. Ho and Penghui Li. *Graded character sheaves, HOMFLY-PT homology, and Hilbert schemes of points on* $\mathbb{C}^2$. May 2, 2023. DOI: 10.48550/arXiv.2305.01306. arXiv: 2305.01306 [math]. URL: http://arxiv.org/abs/2305.01306 (visited on 08/28/2024). Pre-published.

[Zhu25] Xinwen Zhu. *Tame Categorical Local Langlands Correspondence*. Version 1. 2025. DOI: 10.48550/ARXIV.2504.07482. URL: https://arxiv.org/abs/2504.07482 (visited on 04/14/2025). Pre-published.